Modeling of the Soluble Microbial Products (SMP) in Anaerobic Membrane BioReactors (AMBR): Equilibria and stability of the AM2b model [★]

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Abstract: This paper deals with the modification of the AM2 model (Bernard et al. (2001)) to integrate the production and the degradation of the Soluble Microbial Products (SMP) in an Anaerobic Membrane BioReactor (AMBR). The developed model takes into account the characteristics of membrane bioreactors, namely i) there is no biomass in the effluent, ii) part of the SMP, corresponding to the biggest molecules, is retained in the reactional medium, iii) the decay rate and the withdraw of the biomass are considered. Possible equilibria of the system are calculated, and the influence of the maximum growth rate of the biomass on the SMP on these equilibria is analyzed. It is shown that the presence of this new variable in the model changes the qualitative properties of the classical AM2 model.

Keywords: Anaerobic digestion, Biotechnology, Equilibria, Fouling, Membrane BioReactor, SMP modeling.

1. INTRODUCTION

Effluent discharge constraints become more and more stringent and wastewater quantities to be treated increase. It is urgent to conceive high-performance treatment plants capable of rejecting in the natural environment good quality effluents possibly able to be reused in agriculture and/or industry. The technology of Anaerobic Membrane BioReactors (AMBR) appears promising to take up this challenge. However, the risk of membrane fouling limits the development of these systems. Modeling coupling the fouling phenomena together with the biotic characteristics, should allow to make these fouling more predictable and to act by means of a control in order to limit their impact on the system performance.

WWTP effluents contain a large variety of soluble organic compounds: residual substrates, intermediates or final products and Soluble Microbial Products (SMP) (Barker and Stuckey (1999)). The latter regroup organic compounds produced from the substrate metabolism (generally accompanied by a biomass growth) and from the biomass decay during the complete mineralization of simple substrates (Noguera et al. (1994)). Usually two categories of SMP are considered: BAP (Biomass Associated

Products), associated with the biomass mortality, and UAP (Utilization Associated Products), associated with the substrate degradation and the biomass growth (Barker and Stuckey (1999)). The presence of SMP in a MBR affects the process performances (effluent quality) and they can pose problems of toxicity. In addition, their role in the membranes fouling is recognized. It is thus essential to integrate the SMP in the modeling of bioprocesses.

Most MBR are aerobic systems. This is why the efforts to couple biotic phenomena with membrane models have been primarily proposed for the aerobic systems. They are based on the modification of ASM models to include the SMP formation and degradation. On the contrary, in anaerobic conditions, only a few papers are available (Ng and Kim (2007)). Yet, anaerobic digestion systems have many advantages among which the possible use of the biogas produced as an energy source.

The first model to predict the production and the degradation of SMP in an anaerobic chemostat has been proposed by Noguera et al. (1994), in considering two categories of SMP, that are UAP and BAP. The model integrates the UAP production only starting from the initial organic matter (glucose) and considers their degradation only by the acidogenic bacteria. The BAP are considered to be produced by the decay of acidogenic bacteria and slowly degraded by these same bacteria. The model does not take into account neither the UAP and the BAP production starting from the acetate (intermediate product of the

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reaction), nor the decay of methanogenic biomass.

This model was further developed by Barker and Stuckey (2001) who have added i) the concept of the UAP production in consuming both the initial organic matter and the intermediate products by the acidogenic and methanogenic bacteria respectively, ii) the BAP production starting from the decay of acidogenic and methanogenic, and the iii) BAP degradation by acidogenic to grow and give intermediate products.

Latter on, Laspidou and Rittmann (2002) have proposed an unified theory for the production and the degradation of the UAP and the BAP, in considering the coupling between three compounds: Extracellular Polymeric Substances (EPS), SMP and inert biomass. They organize their theory through six hypotheses for how SMP, EPS, active and inert biomass are related.

Recently, Aquino and Stuckey (Aquino and Stuckey (2008)), have refined the model suggested in (Barker and Stuckey (2001)) by taking into account the concept of the EPS formation and degradation as suggested in Laspidou and Rittmann (2002), in order to predict the SMP production.

It should be stressed that the majority of the studies quoted above relate to conventional systems of anaerobic digestion (chemostat-like systems) and not to membrane reactors. In addition, the last developed models are rather complex and, clearly, are not adapted for control system design.

This paper aims at proposing i) a model for an AMBR with external membrane module, which should be simple enough to be used for control purposes and, in the same time, fine enough to allow linking in a pertinent way dynamics of fouling to SMP dynamics. To do so, we propose to modify the AM2 model (Bernard et al. (2001)) and to study equilibria of such a model including the SMP formation and degradation in an AMBR.

This paper is structured as follows. First, we present the model with SMP. Then, it is analyzed and its equilibria are characterized. Finally, simulation results are discussed before conclusions and perspectives are drawn.

2. MATHEMATICAL MODEL

2.1 Hypotheses for the model development

The figure 1 represents the compartments included in the model with S_1 the organic matter concentration (COD), S_{1in} the COD input concentration, S_2 the Volatile Fatty Acid concentration (VFA), S_{2in} the VFA input concentration, X_1 the acidogenic biomass, X_2 the methanogenic biomass and $S = UAP + BAP$, the Soluble Microbial Products (SMP). The model is developed under the following assumptions (H1-H7). The figure 2 represents the matter separation by the membrane which is formalized through assumptions H2, H3 and H6.

Hypothesis 1. The reactional medium of the bioreactor is considered to be a homogeneous.

Hypothesis 2. The substrates S_1 and S_2 cross the external membrane without retention (the molecules size is smaller than the pore diameter).

Hypothesis 3. Total retention of biomasses X_1 and X_2 by the membrane is considered (the bacteria size is greater than the pore diameter). Therefore, there is no solids in

Fig. 1. Schematic representation of the proposed model

the effluent, and thus no term $-DX$ in the mass balance equation of X_i .

Hypothesis 4. Decay rates D_0 of biomasses are taken into account.

Hypothesis 5. The withdraw of the biomass with a flowrate Q_1 is considered.

Hypothesis 6. $SMP = UAP + BAP$ are grouped into an unique variable of concentration S , which only a fraction β leaves the bioreactor. The remaining, corresponding to macromolecules, is retained by the membrane. This is modeled in the mass balance equation by $-\beta DS$ where $0 \leq \beta \leq 1$ ($\beta = 0$: total retention of SMP by the membrane; $\beta = 1$: free crossing of SMP through the membrane).

Hypothesis 7. The bioreactor is supposed to operate in normal conditions (no critical fouling: the flux is lower than the critical flux).

Fig. 2. Matter separation by the membrane

2.2 Reaction schemes and mathematical equations

Our model is based on the modification of the model AM2 that was developed within the framework of the European project AMOCO¹ and discussed in Benyahia et al. (2010). The AM2 model considers only two functions of the anaerobic digestion (Bernard et al. (2001)).

- i- Acidogenesis: the acidogenic bacteria (X_1) consume the organic substrate (S_1) and produce CO_2 and VFA (S_2);

$$k_1 S_1 \rightarrow X_1 + k_2 S_2 + k_4 CO_2$$
- ii- Methanogenesis: the methanogenic bacteria (X_2) use S_2 as substrate to grow, and produce CO_2 and methane CH_4 .

$$k_3 S_2 \rightarrow X_2 + k_5 CO_2 + k_6 CH_4$$

We propose to modify this reaction scheme in adding a new compound SMP named S . We model the SMP production

¹ Acronym for Advanced MONitoring and CONtrol system for anaerobic processes, European FAIR project No. ERB-FAIR-CT96-1198

from the degradation of S_1 , S_2 and the decay of biomasses X_i . In addition, we consider the SMP degradation into S_2 , CH_4 and CO_2 through the growth of X_1 .

- i- Acidogenesis + SMP Production:
 $k_1 S_1 \longrightarrow X_1 + k_2 S_2 + b_3 S + k_4 CO_2$
with reaction rate: $r_1 = \mu_1(S_1)X_1$
- ii- Methanogenesis + SMP Production:
 $k_3 S_2 \longrightarrow X_2 + b_4 S + k_5 CO_2 + k_6 CH_4$
with reaction rate: $r_2 = \mu_2(S_2)X_2$
- iii- SMP Degradation:
 $b_1 S \longrightarrow X_1 + b_2 S_2 + k_7 CO_2$
with reaction rate: $r = \mu(S)X_1$
- iv- SMP Production from biomass decay:
 $X_1 \longrightarrow D_0 S, \quad X_2 \longrightarrow D_0 S$

Let us note $\xi = [S_1, X_1, S_2, X_2, S]^T$ the state space vector of the model AM2+SMP (thereafter, we note this model AM2b). From this new reaction scheme and taking into account the above assumptions (H1-H7), the matter conservation principle enables us to write down the following mathematical model $\dot{\xi} = f(\xi)$.

$$\dot{S}_1 = D(S_{1in} - S_1) - k_1 \mu_1(S_1)X_1 \quad (1)$$

$$\dot{X}_1 = [\mu_1(S_1) + \mu(S) - D_0 - D_1]X_1 \quad (2)$$

$$\dot{S}_2 = D(S_{2in} - S_2) - k_3 \mu_2(S_2)X_2 + [k_2 \mu_1(S_1) + b_2 \mu(S)]X_1 \quad (3)$$

$$\dot{X}_2 = [\mu_2(S_2) - D_0 - D_1]X_2 \quad (4)$$

$$\dot{S} = [b_3 \mu_1(S_1) + D_0 - b_1 \mu(S)]X_1 + [b_4 \mu_2(S_2) + D_0]X_2 - [\beta D + (1 - \beta)D_1]S \quad (5)$$

with:

- k_1 : degradation rate of S_1 by X_1 [g/g]
- k_2 : production rate of S_2 by X_1 from S_1 [mmol/g]
- k_3 : degradation rate of S_2 by X_2 [mmol/g]
- b_1 : degradation rate of S by X_1 [g/g]
- b_2 : production rate of S_2 by X_1 from S [mmol/g]
- b_3 : production rate of S by X_1 from S_1 [g/g]
- b_4 : production rate of S by X_2 from S_2 [g/g]
- β : SMP fraction leaving the bioreactor
- D : dilution rate ($= \frac{Q_{in}}{V}$) [d^{-1}]
- D_0 : decay rate of biomass [d^{-1}]
- D_1 : withdraw of biomass ($= \frac{Q_1}{V}$) [d^{-1}]
- S_{1in} : the COD input concentration (g/l)
- S_{2in} : the VFA input concentration (mmol/l)

3. MODEL EQUILIBRIA, STABILITY AND SIMULATION RESULTS

3.1 Model equilibria

We complete the assumptions of the section 2.1 by the following hypothesis on the model kinetics.

Hypothesis 8. The characteristics of the kinetics μ_1 , μ_2 and μ are:

- $\mu_1(0) = \mu_2(0) = \mu(0) = 0$ (no growth without substrate).
- $\mu'_1(S_1) > 0$ and $\mu'(S) > 0$ for $S_1 \geq 0$ and $S \geq 0$ respectively.
- $\mu_1(\infty) = m_1$ and $\mu(\infty) = m$

- $\mu'_2(S_2) > 0$ if $0 \leq S_2 < S_2^M$
- $\mu_2(S_2)$ reaches a maximum $\mu_2^M = \mu_2(S_2^M)$
- $\mu'_2(S_2) < 0$ for $S_2 > S_2^M$

As examples of such functions, one considers the following Monod functions μ_1 and μ and the Haldane function μ_2 .

$$\mu_1(S_1) := \frac{m_1 S_1}{K_1 + S_1}, \quad \mu(S) := \frac{m S}{K + S},$$

$$\mu_2(S_2) := \frac{m_2 S_2}{K_2 + S_2 + \frac{S_2^2}{K_I}}.$$

To compute the equilibrium points, we cancel the derivatives (1) to (5). From (2) we have $X_1 = 0$ or $\mu_1(S_1) + \mu(S) - D_0 - D_1 = 0$, and from (4), we have $X_2 = 0$ or $\mu_2(S_2) - D_0 - D_1 = 0$. Three cases must be considered.

- $X_1 = 0$. See Lemma 9.
- $X_1 > 0$ and $X_2 = 0$. See Lemma 10.
- $X_1 > 0$ and $X_2 > 0$. See Lemma 11.

Lemma 9. The equilibria (S_1, X_1, S_2, X_2, S) of the system (1-5) for which $X_1 = 0$ are given by

- the washout equilibrium of X_1 and X_2 ,
 $E_0^0 = (S_{1in}, 0, S_{2in}, 0, 0)$, which always exists.
- the washout equilibrium of X_1 but not of X_2 ,
 $E_0^i = (S_{1in}, 0, \lambda_2^i, X_2^i, S^{i*})$, $i = 1$ or 2 , where

$$X_2^i = \frac{D(S_{2in} - \lambda_2^i)}{k_3(D_0 + D_1)},$$

$$S^{i*} = \frac{b_4 + \frac{D_0}{D_0 + D_1}}{[\beta + (1 - \beta)\frac{D_1}{D}]k_3}(S_{2in} - \lambda_2^i), \quad i = 1, 2$$

Which exists if and only if $S_{2in} > \lambda_2^i$.

In the sequel we use the following notations. Let

$$F(S) := \mu_1^{-1}(D_0 + D_1 - \mu(S)) \quad (6)$$

$$G(S_1) := \frac{1}{B}[S_{1in} - S_1][\frac{1}{k_1}(b_3 + b_1) + \frac{1}{k_1 \mu_1}(D_0 - b_1(D_1 + D_0))]$$

$$H_i(S_1) := \frac{1}{B}[A(S_{2in} - \lambda_2^i) + (S_{1in} - S_1)(\frac{A_1 - A_2}{k_1} + \frac{A_2(D_1 + D_0) + D_0}{k_1 \mu_1})], \quad i = 1, 2 \quad (8)$$

Where $S_2 = \lambda_2^i$ is a solution of equation $\mu_2(S_2) = D_0 + D_1$ and,

$$\begin{cases} A = \frac{b_4(D_0 + D_1) + D_0}{k_3(D_0 + D_1)} \\ A_1 = b_3 + A k_2, \quad A_2 = A b_2 - b_1 \\ B = \beta + (1 - \beta)\frac{D_1}{D} \end{cases} \quad (9)$$

Lemma 10. Let (X_1, X_2, S_1, S_2, S) an equilibrium point of system (1-5). If $X_1 > 0$ and $X_2 = 0$ then one has $0 < S_1 < S_{1in}$, $S > 0$ and $S_2 > 0$. Moreover S_1 and S are solutions of equations system:

$$\begin{cases} S_1 = F(S) \\ S = G(S_1) \end{cases} \quad (10)$$

and X_1 and S_2 are given by the formulas

$$X_1 = D \frac{S_{1in} - S_1}{k_1 \mu_1},$$

$$S_2 = S_{2in} + [k_2 \mu_1 + b_2 \mu] \frac{S_{1in} - S_1}{k_1 \mu_1}.$$

Lemma 11. Let (X_1, X_2, S_1, S_2, S) an equilibrium point of system (1-5). If $X_1 > 0$ and $X_2 > 0$ then one has $0 < S_1 < S_{1in}$, $S > 0$ and $S_2 = \lambda_2^i$ ($i = 1, 2$). Moreover S_1 and S are solutions of equations system:

$$\begin{cases} S_1 = F(S) \\ S = H_i(S_1), \quad i = 1, 2 \end{cases} \quad (11)$$

and X_1 and X_2 are given by the formulas

$$X_1 = D \frac{S_{1in} - S_1}{k_1 \mu_1},$$

$$X_2 = D \frac{[S_{1in} - S_1][k_2 \mu_1 + b_2 \mu] + (S_{2in} - \lambda_2^i) k_1 \mu_1}{k_1 k_3 (D_0 + D_1) \mu_1}.$$

With the following condition:

$$\lambda_2^i < S_{2in} + (k_2 \mu_1 + b_2 \mu) \frac{S_{1in} - S_1}{k_1 \mu_1}.$$

3.2 Graphic calculation of equilibria

The equilibrium points are obtained from the graphic calculation of the coordinates (S_1, S) of the intersections of the graph of function F with the graphs of functions G , H_1 and H_2 .

The F graph depends on the value of maximum growth rate m of $\mu(S)$. It takes the form of vertical line when $m = 0$, then becomes deformed in hyperbolic branch towards the left by increasing the value m (figure 3.2). The functions $H_1(S_1)$, $H_2(S_2)$ and $G(S_1)$ do not depend on the kinetic $\mu(S)$. Their graphs do not change when m is changed. These functions can be written in the form:

$$S = G(S_1) = (S_{1in} - S_1) \left(C'_1 + \frac{C'_2}{\mu_1(S_1)} \right).$$

$$S = H_i(S_1) = C_1 + (S_{1in} - S_1) \left(C_2 + \frac{C_3}{\mu_1(S_1)} \right).$$

Where:

$$C'_1 = \frac{b_3 + b_1}{k_1 B}, \quad C'_2 = \frac{D_0 - b_1(D_0 + D_1)}{k_1 B},$$

$$C_1 = \frac{A}{B}(S_{2in} - \lambda_2^i), \quad C_2 = \frac{A_1 - A_2}{k_1 B},$$

$$C_3 = \frac{A_2(D_0 + D_1) + D_0}{k_1 B}.$$

Before studying the graphical forms of the functions, we present the following inequalities which are imposed by the biological mass balance principle.

$$k_1 \geq 1 + b_3 + k_2 \quad (12)$$

$$k_3 \geq 1 + b_4 \quad (13)$$

$$b_1 \geq 1 + b_2 \quad (14)$$

This expresses the fact that the quantity of biomass and products produced is always smaller than the quantity of substrate consumed.

Since the SMP are slowly produced and degraded in the bioreactor, we can write,

- $k_1 > b_1$: The degraded quantity S of SMP is smaller than the degraded quantity S_1 of substrate.
- $k_2 > b_2$: The produced quantity S_2 of VFA from S_1 is higher than the produced quantity from the SMP.
- $k_2 > b_3$: The produced quantity S_2 of VFA from S_1 is higher than the produced quantity S of SMP from S_1 .
- $b_4 < k_1, k_2, k_3$: The produced quantity S of SMP from S_2 is small, the most part of S_2 is converted in biogas.

Because $k_2 > b_2$, we have:

$$C_2 > 0,$$

and from (13), we can deduce easily that the yield

$$A < 1,$$

which confirms a biological principle (A given by (9) represents the quantity of SMP produced on the quantity of S_2 degraded). Now, from (14) and with the inequality $A < 1$, we conclude that $b_1 > Ab_2 + \frac{D_0}{D_0 + D_1}$, which forces

$$C_3 < 0.$$

Since $0 < S_1 < S_{1in}$ (no washout), and under the conditions $C'_1 > 0$, $C'_2 < 0$, $C_2 > 0$, $C_3 < 0$ and $C_3 > -m_1 C_2$, there exists sub-domains of $S_1 \in]0, S_{1in}[$, such that $S = G(S_1)$ and $S = H_i(S_1)$ are positive and are limited (we have a limited production of S).

The graphs of the functions G , H_1 and H_2 are as it is shown in figure 3.

We note on all the following figures:

$$\lambda_1 = \mu^{-1}(D_0 + D_1).$$

$S_1^{H_1=0}$: the greatest value of S_1 such that, $H_1(S_1^{H_1=0}) = 0$.

$S_1^{H_2=0}$: the greatest value of S_1 such that, $H_2(S_1^{H_2=0}) = 0$.

We remark from the formula (7) that $G(S_{1in}) = 0$.

S_{1in} can take three positions compared to $S_1^{H_1=0}$ and $S_1^{H_2=0}$.

- Case 1: $S_{1in} \leq S_1^{H_2=0} < S_1^{H_1=0}$.
- Case 2: $S_1^{H_2=0} < S_{1in} \leq S_1^{H_1=0}$.
- Case 3: $S_1^{H_2=0} < S_1^{H_1=0} < S_{1in}$.

Fig. 3. The position of S_{1in} compared to $S_1^{H_2=0}$ and $S_1^{H_1=0}$

For illustration, we present on the figure 3 the case 1.

We will analyse in the following the case 2, the other cases can be analyzed in the same way (this case is given as example according to the selected set of parameters of table 1).

We trace all the graphs F , G and H_i together and we look

at the possible positions of λ_1 compared to S_{1in} , $S_1^{H_1=0}$ and $S_1^{H_2=0}$, by changing only one parameter of the model, the other parameters being fixed. We change K_1 and we consider three sub-cases as it is shown by the figures 3.2, 3.2 and 3.2 (other cases can exist according to the selected biological parameters).

- Case 2.1: $\lambda_1 < S_1^{H_2=0} < S_{1in} < S_1^{H_1=0}$.
- Case 2.2: $S_1^{H_2=0} < \lambda_1 < S_{1in} < S_1^{H_1=0}$.
- Case 2.3: $S_1^{H_2=0} < S_{1in} < S_1^{H_1=0} < \lambda_1$.

The intersections of the graphs F with G and H_i give the solution (S_1, S) of the equations systems (6-7) and (6-8) and thereafter, allow to calculate the equilibria according to the formulas of lemmas 10 and 11.

Fig. 4. The intersections of F with G , H_i in the case 2.1

Fig. 5. The intersections of F with G , H_i in the case 2.2

Fig. 6. The intersections of F with G , H_i in the case 2.3

3.3 Simulation results and discussions

To illustrate our approach, we perform simulations with nominal parameter values given by the table 1.

The parameters of the model part without SMP (initial model AM2) have values equal or close to the AMOCO values (Bernard et al. (2001)). The other parameters of the part modeling the SMP, have values satisfying the below

inequalities between k_i and b_i and, the inequalities (12), (13) and (14) (by transforming the coefficients units in g/g).

Table 1. Nominal parameters values

Parameters	Nominal values	Parameters	Nominal values
m_1	1.2	k_3	268
K_1	change	b_1	5
m_2	1.5	b_2	10
K_2	5	b_3	7
K_I	15	b_4	5
β	0.6	m	[0..1]
k_1	25	K	3
k_2	250		

The values of the operating parameters are.

$$D = 1, D_0 = 0.25, D_1 = 0.25, S_{1in} = 10, S_{2in} = 10$$

We are interested in this paper in equilibria of the case shown by the figure 3.2 in which we have three equilibria. For $m = 0$, the function F is a vertical line on the left of S_{1in} and intersects the functions G and H_i in three points, which gives the same solution λ_1 on the x -axis and three corresponding solutions S^* on the y -axis. In fact, for $m = 0$, we have exactly the case of the AM2 model with three equilibria in the phase plans (S_1, X_1) and (S_2, X_2) as it is shown by the figure 7 on the left. In this case, X_1 has the same values.

Fig. 7. The phase plans (S_1, X_1) and (S_2, X_2) for $m = 0$ (left), $m = 0.27$ (center) and $m = 0.8$ (right)

For increasing values of m , the X_1 concentration increases because the S_1 concentration decreases. In the cases of equilibria E_{21}^1 and E_{21}^2 , the X_2 concentration increases if m increases and in the case of the washout equilibrium E_{21}^0 , it is S_2 concentration which increases with m (figure 7 on the center and on the right). The biomasses growth on the SMP can modify the system equilibria.

To study the stabilities of these equilibria, we calculated the Jacobian of system f given by (1-5) and we replaced equilibria by their numerical values calculated from the graphs intersections. We calculated the eigenvalues for each equilibrium. Equilibria and stability are given by the following theorem.

Theorem 12. In the case 2.1, the system (1-5) can have three equilibria:

- $E_{21}^1 = (\lambda_{21}^1, X_{1,21}^1, \lambda_{21}^2, X_{2,21}^1, S_{21}^1) : \text{Stable}$

- $E_{21}^2 = (\lambda_{21}^2, X_{1,21}^2, \lambda_{21}^2, X_{2,21}^2, S_{21}^2) : \text{Unstable}$
- $E_{21}^0 = (\lambda_{21}^0, X_{1,21}^0, S_{2,21}^0, 0, S_{21}^0) : \text{Stable}$

Where:

$(\lambda_{21}^0, S_{21}^0)$ and $(\lambda_{21}^1, S_{21}^1)$, $(\lambda_{21}^2, S_{21}^2)$ are the solutions of equations systems (10) and (11) respectively.

$(X_{1,21}^0, X_{1,21}^1, X_{1,21}^2)$, and $(X_2 = 0, X_{2,21}^1, X_{2,21}^2)$ are the values of X_1 and X_2 given by the lemma 10 and 11. $(S_{2,21}^0)$ is the quantity of S_2 given by the lemma 10.

We can also note in the figure 7 that equilibria E_{21}^1 and E_{21}^0 are attractive and locally stable, while equilibrium E_{21}^2 is unstable.

According to these results, we conclude that the original system AM2 perturbed by the introduction of SMP keeps the number and the nature of its equilibria. Only their values change with m . For values of m close to 1 (sufficiently larger than 0), we have high concentrations of X_1 and X_2 , lower concentrations of S_1 , S_2 and S , which is important in this case. But, let us note in the figure 3.2 that for such values of m , the equilibria E_{21}^1 and E_{21}^2 approach and thus the attraction basin of the operational equilibrium is restricted, which weakens system stability. It is important from practical point of view to synthesize control laws stabilizing the system around the operational positive equilibrium E_{21}^1 .

We remind that according to the biological parameters used, we can have other cases of the positions of the numbers λ_1 , S_{1in} , $S_1^{H_2=0}$ and $S_1^{H_1=0}$ and thus the number and the nature of equilibria changes according to each case. For example, in the case of the figure 3.2, we can have nine equilibria of which four are stable, the system has a quadri-stability.

3.4 SMP influence on the equilibrium λ_1 of acidogenesis

We noted that in absence of the growth on the SMP, i.e. when $m = 0$, the AM2b model behaves exactly as the AM2 model. If the input organic matter S_{1in} changes, then the value of equilibrium λ_1 remains unchangeable as it is shown by the figure 3.4 (we present only the graphs of F and H_1).

Fig. 8. Equilibrium λ_1 according to S_{1in} for $m = 0$ and $m = 0.1$

For the case $m \neq 0$, we remark that if S_{1in} increases, then the value of λ_1 decreases. This shows that the SMP integration in the model, can predict the change of equilibrium λ_1 , which can occur, when the influent concentration S_{1in} of the organic matter changes. In fact, the equilibria λ_1 of the AM2b model are calculated by the resolution of (10) and (11), which contain explicitly the parameter m in the function $F(S)$. On the contrary in the

AM2 model, there is no information on the SMP in the expression of equilibrium λ_1 .

4. CONCLUSION

This paper presents the development and the analysis of a mathematical model for an AMBR. The model is based on the modification of the AM2 model of anaerobic digestion proposed in Bernard et al. (2001) and analyzed in Benyahia et al. (2010). Under general assumptions, it is shown that the modified AM2 model can keep the number and the nature of its equilibria. However, under certain operating conditions or if biological parameters values other than the default ones are used, the nature of the AM2b equilibria changes. Perspectives of the work include the characterization of the qualitative properties of the new proposed model, notably when the model exhibits multi-stability property.

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