Homogeneous polynomial Lyapunov functions for the analysis of polytopic parameter-dependent descriptor systems

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Abstract—In this paper, we present an extension of the homogeneous polynomial Lyapunov functions (HPLFs) approach previously developed for uncertain standard systems to the case of uncertain continuous-time descriptor ones. By employing a new power transformation of the state vector with respect to its dynamic and algebraic parts, new necessary and sufficient admissibility analysis conditions are developed based on such a class of Lyapunov functions. The presented results deal with time-varying parameters with both unbounded variation rate, based on parameter-independent HPLFs, and bounded variation rates lying inside a polytope by exploiting polytopic parameter-dependent HPLFs. Numerically well tractable sufficient LMI conditions are also derived for both cases. A comparison with results from the literature is performed, showing the reduction of conservatism throughout the novel conditions presented in this article.

I. INTRODUCTION

Uncertain descriptor systems have been a topic of active research because they can represent altogether a comprehensive set of phenomena, such as time-varying dynamics, impulsive behavior and algebraic constraints. It is commonly found in the literature that this mathematical representation serves to fields such as robotics, aeronautics, economics, electronics, among others, according to [1], [2], [3].

Lyapunov theory based on the notion of Lyapunov functions (LFs) is very appealing for the analysis and control of complex systems since LFs can be interpreted as an energy function and their generic formulation can be used along with many different classes of systems. Lyapunov functions have been massively employed in order to develop mathematical tools for the admissibility analysis and control synthesis of uncertain descriptor systems as they can provide non conservative stability analysis criteria, according to [4], [5].

The absence of conservatism is related to the concept of universality between a class of Lyapunov functions and a class of systems. As defined in [6], a given class $\mathcal{A}$ of LFs is said to be universal with respect to a class $\mathcal{B}$ of systems if the stability of system in $\mathcal{B}$ is equivalent to the existence of a LF in class $\mathcal{A}$. Notice that quadratic Lyapunov functions are universal for standard linear time-invariant (LTI) systems, yielding to necessary and sufficient stability conditions that can be easily translated into linear matrix inequalities (LMIs). However, the universality of quadratic LFs no longer holds for more complex classes of dynamical systems such as the uncertain systems class. Indeed, it has been shown

in [7], [4], [6], [8] that quadratic LFs provide conservative results when employed for the analysis of uncertain systems. This motivated the study of different classes of LFs, such as polyhedral [6], homogeneous polynomial [7], [8], [5] and biquadratic [9], [10].

Homogeneous polynomial Lyapunov functions (HPLFs) constitute a universal class for stability analysis of uncertain parameter-dependent systems [7], [8], [5]. Since HPLFs are constructed based on a power transformation of the state vector, it has been shown in [7], [8] that the stability analysis of uncertain parameter-dependent systems based on HPLFs can be reformulated as a quadratic stability analysis problem of an associated extended system. Due to this fact, in [8], robust LMI stability analysis conditions have been derived by exploiting the degree of freedom related to the complete square matricial representation (CSMR) of HPLFs. Moreover, it has been shown that HPLFs provide necessary and sufficient conditions for stability analysis once a suitable degree for the homogeneous polynomials is chosen. Notice that much of the work done with regards to the stability analysis of standard uncertain systems using HPLFs is yet to be transposed to the uncertain descriptor case. This is not straightforward since it raises the difficult question of how the power transformation should be applied with respect to the algebraic state apart from the dynamic state of the descriptor system. Indeed, this paper aims to provide a contribution not only in transferring techniques from the standard to the descriptor framework but also in supplying numerically exploitable, less restrictive, admissibility conditions to the latter.

Among the works related to the admissibility analysis of uncertain descriptor systems, the commonly employed Lyapunov functions are quadratic with respect to the states and either affine/polytopic parameter-dependent ([11], [9], [10], [12]) or quadratic parameter-dependent ([9], [10]). Each of these results use slack variables to decouple Lyapunov and state matrices. Among the derived LMI conditions that have affine/polytopic parameter dependence, [11], [9], [10] use parameter-independent slack variables, while [12] employs polytopic slack variables and sum-of-squares polynomials. The LMI conditions obtained in the context of quadratic parameter dependence ([9], [10]) use quadratic parameter-dependent slack variables. As far as we know, the only results available in the literature introducing HPLFs for the admissibility analysis of uncertain descriptor systems have been presented in our previous work [13]. In spite of using parameter-independent Lyapunov functions, the numerical examples show that the use of HPLFs may lead to less

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restrictive results.

In this paper, we present new results on the admissibility analysis of continuous-time uncertain descriptor systems by using parameter-independent and parameter-dependent HPLFs. First, by reviewing our previous work [13], we present necessary and sufficient conditions for the existence of a parameter-independent HPLF of a given degree that evaluates the admissibility of a uncertain descriptor system with unbounded variation rate of the parameters. Numerically tractable LMI conditions are also derived. Then, the admissibility of uncertain descriptor systems is assessed through parameter-dependent HPLFs as an extension of the parameter-independent class. Necessary and sufficient conditions are presented and, subsequently, two sets of LMI conditions are derived. These novel conditions for parameter-dependent HPLFs are obtained by two different convexification techniques. Overall, the results presented in this paper extend the results proposed by [7], [8] for uncertain standard systems to the case of uncertain descriptor ones. This is achieved by considering the descriptor system under its SVD normal form and by delivering an explicit formulation of the power transformation, clearly emphasizing the dependence on dynamic and algebraic states of the descriptor system, as well as of the associated extended descriptor system.

The paper is organized as follows: section II presents the problem and previously developed mathematical tools; section III brings the main contributions of this paper; section IV shows reduced conservatism of the proposed approach on numerical examples; and section V summarizes and concludes the manuscript.

Notations: The notation used in this manuscript is standard. The superscript T over a vector or matrix stands for its transposition. The symbol $(\star)$ in a matrix indicates a block induced by symmetry. For a vector w , w_i stands for its i -th component. The convex hull generated by the vectors $w^{(1)}, \dots, w^{(N)}$ is given as $\text{Co}(w^{(1)}, \dots, w^{(N)})$. The sum between a matrix and its transposed version is indicated by $\text{He}\{A\} = A + A^T$ and a matrix composed by diagonal appendages of other matrices is given as $\text{Bdiag}\{A, B\} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. The Kronecker product between given matrices A and B is noted as $A \otimes B$. Scalars as subscripts indicate the dimensions of matrices *e.g.* 0_n stands for a zero square matrix in $\mathbb{R}^{n \times n}$ while $0_{n \times n_\xi}$ represents a zero rectangular matrix in $\mathbb{R}^{n \times n_\xi}$. The degree of the polynomial expression S is indicated by $\deg(S)$. For two given sets A and B , $A \times B$ represents the cartesian product between them. The infinity norm of a vector $d \in \mathbb{R}^n$ is given by $\|d\|_\infty = \max\{|d_1|, \dots, |d_n|\}$. The orthogonal complement of a square matrix $E \in \mathbb{R}^{(n+n_\xi) \times (n+n_\xi)}$ of rank n is the matrix $E_0 \in \mathbb{R}^{(n+n_\xi) \times n_\xi}$ such that $E^T E_0 = 0_{(n+n_\xi) \times n_\xi}$.

II. PROBLEM STATEMENT AND PRELIMINARIES

The current section presents the system of interest and recollects mathematical tools that are used in the main contribution of this paper.

A. Polytopic parameter-dependent descriptor systems

Consider parameter-dependent continuous-time descriptor systems in the singular value decomposition (SVD) normal form

$$\begin{cases} \dot{x}(t) = A_{11}(w(t))x(t) + A_{12}(w(t))\xi(t) + B_1(w(t))u(t) \\ 0_{n_\xi \times 1} = A_{21}(w(t))x(t) + A_{22}(w(t))\xi(t) + B_2(w(t))u(t) \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the dynamic state, $\xi(t) \in \mathbb{R}^{n_\xi}$ is the algebraic state and $u(t) \in \mathbb{R}^{n_u}$ is the control input. The initial condition $(x(0), \xi(0)) = (x_0, \xi_0)$ is assumed to be consistent in the sense defined later in the paper.

The time-varying parameters $w(t) \in \mathbb{R}^m$ lie within the unit simplex and may have bounded rates of variation. Therefore,

$$w(t) \in \mathcal{W} = \{w \in \mathbb{R}^m \mid \sum_{i=1}^m w_i = 1, 0 \leq w_i \leq 1\} \quad \forall t \geq 0.$$

The matrices $A_{ij}(w(t))$, for $i, j \in \{1, 2\}$, and $B_i(w(t))$, for $i \in \{1, 2\}$, are described within polytopes of matrices as

$$\begin{aligned} A_{ij}(w(t)) &= \sum_{k=1}^m w_k(t) A_{ij}^{(k)} \\ B_i(w(t)) &= \sum_{k=1}^m w_k(t) B_i^{(k)} \end{aligned} \quad (2)$$

where vertex matrices $A_{ij}^{(k)}$ and $B_i^{(k)}$ are known real-valued matrices of appropriate dimension. Notice that polytopic linear models have been widely used in the literature since they naturally arise as a description of uncertain systems or as an approximate and alternative description of nonlinear dynamical systems. For instance, this formalism has been employed for describing, to cite a few, rotational robotic manipulator systems [3], power systems [14] and aircraft dynamic systems [15] with multiple operating conditions.

For many physical systems, bounds on the variation rate of the parameters may be deduced from physical considerations [15]. In the sequel, as already considered by [16], [5], [17] for continuous-time systems and by [18] for discrete-time ones, we assume that the parameters velocity space is specified as a polyhedral convex set *i.e.* $\dot{w}(t) \in \mathcal{V} \quad \forall t \geq 0$ where

$$\mathcal{V} = \{v \in \mathbb{R}^m, v \in \text{Co}(d^{(1)}, \dots, d^{(h)}) \mid \sum_{i=1}^m d_i^{(j)} = 0, j = 1, \dots, h\}. \quad (3)$$

with $d^{(j)}$, for $j = 1, \dots, h$, given vectors. This definition ensures that the constraint $\sum_{i=1}^m \dot{w}_i(t) = 0$ always holds considering that $w(t)$ belongs to the unit simplex $\mathcal{W}$. The interested reader can refer to [16], [5], [18], [19] for detailed procedures allowing to obtain the vectors $d^{(j)}$ from known bounds on the parameters variation rate. The fact that $\dot{w}(t) \in \mathcal{V}$ implies the following polytopic description for the parameters variation rate:

$$\dot{w}(t) = \sum_{j=1}^h \sigma_j(t) d^{(j)} \quad (4)$$

where $\sigma(t) \in \{\sigma \in \mathbb{R}^h \mid \sum_{i=1}^h \sigma_i = 1, 0 \leq \sigma_i \leq 1\} \quad \forall t \geq 0$.

The initial condition (x_0, ξ_0) has to satisfy the algebraic constraint

$$A_{21}(w(0))x_0 + A_{22}(w(0))\xi_0 + B_2(w(0))u(0) = 0.$$

Given the polytopic structure described in (2) of system state matrices and since $w(0) \in \mathcal{W}$, it follows that this constraint is equivalent to the same type of constraint evaluated on the vertices $w^{(i)}$ of the unit simplex $\mathcal{W}$. Therefore, the initial condition (x_0, ξ_0) for system (1) is called *consistent* if the following conditions hold

$$A_{21}^{(i)}x_0 + A_{22}^{(i)}\xi_0 + B_2^{(i)}u(0) = 0 \text{ for } i = 1, \dots, m.$$

We recall that a descriptor system in the general form

$$E\dot{X}(t) = A(t)X(t), \quad (5)$$

with $X \in \mathbb{R}^{(n+n_\xi)}$, $\text{rank}(E) = n \leq n + n_\xi$ and E being a constant matrix, can be converted into the SVD normal form (1) through a constant, but not unique, coordinate transformation. As a matter of fact, the conversion can be done either by rank decomposition, according to [20], or singular value decomposition on the matrix E , as presented by [21], [22].

For systems in the form

$$E\dot{X}(t) = A(w(t))X(t) \quad (6)$$

with constant derivative matrix E , $w(t) \in \mathcal{W}$ and $\dot{w}(t) \in \mathcal{V}$, an important definition follow.

Definition II.1: The pair $(E, A(w(t)))$ is:

- *Regular* if $\det(sE - A(w(t))) \neq 0 \forall w(t) \in \mathcal{W}$ and for some $s \in \mathbb{C}$, indicating existence and unicity of solutions;
- *Impulse-free* if $\deg(\det(sE - A(w(t)))) = \text{rank } E \forall w(t) \in \mathcal{W}$, ensuring the absence of impulsive modes;
- *Robustly stable* if the finite modes are stable for all possible values and trajectories of parameters in the admissible parameter domain;
- *Robustly admissible* if it is regular, impulse-free and robustly stable for all possible values and trajectories of parameters in the admissible parameter domain.

Remark II.1: The robust admissibility property of continuous-time uncertain descriptor system (6) can be expressed through its equivalent SVD normal form (1): regularity and impulsive-free properties are equivalent to the invertibility of the matrix $A_{22}(w(t))$ for all possible values of the vector of parameters $w(t)$; the stability of the finite modes is given by the stability of the equivalent standard form $\dot{x}(t) = (A_{11}(w(t)) - A_{12}(w(t))A_{22}^{-1}(w(t))A_{21}(w(t)))x(t)$ for all possible values and trajectories of parameters in the admissible parameter domain.

Remark II.2: The descriptor system in the general form (6), leading to form (1) through SVD decomposition, does not exhibit parameter dependence on the derivative matrix E . In fact, in the case of *constant uncertain parameters*, a constant E matrix is not restrictive according to [23], [9] since an augmented descriptor system with constant derivative matrix can be obtained whenever the matrices E and A are affine

parameter-dependent. Although this technique was proposed for discrete-time systems, it easily generalizes to continuous-time ones. In the case of *time-varying parameters*, few approaches have been proposed for the robust admissibility analysis of descriptor systems with parameter dependence in the derivative matrix E . For instance, the results of [24], [25], [26] have been developed for discrete-time systems. The aim of our work is to generalize the HPLF framework developed for regular (standard) systems to the case of singular (descriptor) ones. Notice that the HPLF approach cannot easily be broadened to discrete-time parameter-dependent systems since the state power transformation, presented in the next section, generates a nonlinear parameter dependence on the extended system that is very difficult to deal with. Although [27] addressed continuous-time uncertain descriptor systems in addition to discrete-time ones, the proposed results, alike the aforementioned ones, use the quadratic Lyapunov functions framework that may be more conservative than the HPLF one as shown by our numerical examples. In our paper, the generalization of the HPLF framework to the case of descriptor systems has been reached by proposing an explicit formulation of the state power transformation according to the dynamic and algebraic parts of the state vector of the original system. Handling general parameter-dependent derivative matrix E in the HPLF framework is still a difficult problem and yet to be investigated.

In the following, we present the main concepts related to the HPLF framework.

B. Homogeneous polynomial Lyapunov functions

In the literature, HPLFs have been investigated for the robust stability analysis of uncertain standard systems (*i.e.* $E = I_n$) in [8], [28], [29], [5]. The main motivation in using this class of Lyapunov functions for the analysis of uncertain systems is that HPLFs can lead to non conservative conditions once a suitable polynomial degree is chosen.

In the sequel, a compilation of important notions related to HPLFs is recalled.

Definition II.2 (homogeneous forms): The vector $x^{[q]} \in \mathbb{R}^{d_q}$ denotes the *base vector of homogeneous forms of degree q* in $x \in \mathbb{R}^n$. The base vector of homogeneous forms $x^{[q]}$ is a collection of all the integer powered monomials of degree q that can be produced from the entries of the vector x . The components of $x^{[q]}$ are given as

$$x_l^{[q]} = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, \quad i_1 + i_2 + \cdots + i_n = q, \\ i_1, i_2, \dots, i_n \geq 0, \text{ for } l \in \{1, \dots, d_q\},$$

d_q is the dimension of $x^{[q]}$ given by $d_q = \frac{(n+q-1)!}{(n-1)!(q)!}$.

The mapping between x and $x^{[q]}$ stated in Definition II.2 is referred in the literature as the *power transformation of degree q of x* (see [30], [31]).

Definition II.3 (extended matrix): ([30], [31], [7], [8]) Let $\dot{x} = Ax$. The *extended matrix* $A_{[q]} \in \mathbb{R}^{d_q \times d_q}$ associated to the

dynamic of the base vector of homogeneous forms of degree q is given as

$$\frac{d}{dt}x^{[q]} = \frac{\partial x^{[q]}}{\partial x}Ax = A_{[q]}x^{[q]}.$$

Definition II.4 (*extended system*): Consider a parameter-dependent continuous-time standard state-space system

$$\dot{x}(t) = \left(\sum_{i=1}^m w_i(t)A_i \right)x(t), \quad (7)$$

and denote by $A_{i[q]}$ the extended matrix of A_i , according to Definition II.3. Then, the *extended system* associated to the base vector of homogeneous forms of degree q is defined as:

$$\begin{aligned} \frac{d}{dt}x^{[q]}(t) &= A(w(t))_{[q]}x^{[q]}(t) \\ &= \left(\sum_{i=1}^m w_i(t)A_{i[q]} \right)x^{[q]}(t). \end{aligned} \quad (8)$$

This last definition specifies that in the case of continuous-time parameter-dependent systems, the associated extended system (8) preserves the linear parameter dependence of the original system (7). Note that this is not the case for discrete-time systems.

Lemma II.1: [32] There exist matrices M_i , N_i , for $i = 1, \dots, d_{q-1}$, such that the extended matrix $A(w(t))_{[q]}$ is given by

$$A(w(t))_{[q]} = \sum_{i=1}^{d_{q-1}} M_i A(w(t)) N_i, \quad (9)$$

where $d_{q-1} = \dim(x^{[q-1]}) = \frac{(n+q-2)!}{(n-1)!(q-1)!}$, and $M_i \in \mathbb{R}^{d_q \times n}$, $N_i \in \mathbb{R}^{n \times d_q}$ are constant matrices depending only on the order of monomials of degree q stacked in the vector $x^{[q]}$.

Definition II.5 (*homogeneous polynomial Lyapunov functions*): HPLFs of degree $2q$ are scalar functions given by weighted sums of the entries of the base vector of homogeneous forms of degree $2q$. They can be described through the square matricial representation (SMR), given by

$$V_{2q}(x) = x^{[q]T} P x^{[q]}. \quad (10)$$

The matrix $P = P^T \in \mathbb{R}^{d_q \times d_q}$ is non-unique for a given HPLF. Then, consider the linear space

$$\mathcal{P} = \{P_0 = P_0^T \in \mathbb{R}^{d_q \times d_q} \mid x^{[q]T} P_0 x^{[q]} = 0 \ \forall \ x \in \mathbb{R}^n\} \quad (11)$$

whose size is, according to [5]:

$$d_{\mathcal{P}} = \frac{1}{2}d_q(d_q + 1) - \frac{(n+2q-1)!}{(n-1)!(2q)!}.$$

Therefore, the complete square matricial representation (CSMR) of the Lyapunov function (10) is

$$V_{2q}(x) = x^{[q]T} (P + \Gamma(\gamma)) x^{[q]} \quad (12)$$

where $\gamma \in \mathbb{R}^{d_{\mathcal{P}}}$ and matrix $\Gamma(\gamma)$ is a linear parameterization of $\mathcal{P}$ such as $\Gamma(\gamma) = \sum_{i=1}^{d_{\mathcal{P}}} \gamma_i P_{0i}$, $P_{0i} \in \mathcal{P}$.

Remark II.3: The power transformation of degree q of x stated in Definition II.2 along with Definition II.5 are particularly important since they lay the foundations of the HPLF approach. Indeed, the work of [7] showed that a parameter-dependent system (7) is globally asymptotically stable if and only if the extended system (8) is also globally asymptotically stable. This is due to the fact that the time-derivative of the Lyapunov function $V_{2q}(x)$ given by (10) along the trajectories of the uncertain system (7) equals the time-derivative of $V_{2q}(x)$ along the trajectories of the extended system (8) meaning that $\left. \frac{d}{dt}V_{2q}(x) \right|_{(7)} = \left. \frac{d}{dt}V_{2q}(x) \right|_{(8)}$ as demonstrated by [7], [8]. This shows that the power transformation defined in Definition II.2 allows to reformulate the problem of finding a HPLF of degree $2q$, as defined in Definition II.5, for an uncertain system as an equivalent problem of finding a quadratic LF for the extended system.

III. MAIN RESULTS

Based on the concept of HPLFs used for standard uncertain systems and summarized in the previous section, the aim of this section is to extend the results proposed by [7], [8] for uncertain standard systems to the case of uncertain descriptor ones. It is important to notice that the extension of the HPLF framework from standard systems to singular ones is not straightforward since it raises the difficult question of how the base vector of homogeneous forms should be constructed with regard to the dynamic and algebraic states of the descriptor system. In other words, unlike regular systems, the key question is how the power transformation should deal with the algebraic state apart from the dynamic state of the descriptor system. In the following, we address this question by considering the descriptor system under the SVD normal form and by presenting an explicit formulation of the power transformation, clearly emphasizing the dependence on dynamic and algebraic states of the descriptor system, as well as of the associated extended descriptor system.

In the sequel, we propose new admissibility analysis conditions for continuous-time uncertain descriptor systems by means of HPLFs of degree $2q$. Firstly, we tackle the case of parameters with arbitrary rate of variation by using parameter-independent HPLFs. Secondly, we address the case of parameters with bounded rates of variation by employing polytopic parameter-dependent HPLFs. For both cases, we introduce necessary and sufficient conditions for the robust admissibility analysis of uncertain descriptor systems, and then, we derive numerically tractable sufficient LMI conditions.

A. Admissibility analysis with parameter-independent HPLFs

This section presents parameter-independent HPLFs for the admissibility analysis of continuous-time uncertain descriptor systems subject to time-varying parameters with unbounded variation rates. This generalizes the results of [7], [8] to the descriptor systems case and has been introduced

for the first time in our previous work [13]. The proposed method is reviewed in the following.

Based on Definition II.1 and Remark II.1 in section II-A, we state the following definition.

Definition III.1: The uncertain descriptor system under SVD normal form (1) is robustly admissible with a parameter-independent HPLF of degree $2q$ if the matrix $A_{22}(w(t))$ is invertible, for all possible values of the vector of parameters $w(t) \in \mathcal{W}$ and its equivalent standard state-space system

$$\begin{aligned} \dot{x}(t) &= \Gamma(w(t))x(t), \quad \text{where} \\ \Gamma(w(t)) &= A_{11}(w(t)) - A_{12}(w(t))A_{22}(w(t))^{-1}A_{21}(w(t)) \end{aligned} \quad (13)$$

is robustly stable based on a parameter-independent HPLF of degree $2q$, for all possible parameters trajectories of $w(t) \in \mathcal{W}$.

The next lemma generalizes the HPLF approach to the descriptor systems case and gives an explicit formulation of the base vector of homogeneous forms of degree q , according to the dynamic and algebraic states of the descriptor system, as well as of the extended descriptor system. This lemma, stated as a sufficient condition in our previous work [13], is reviewed as a necessary and sufficient condition as follows.

Lemma III.1: The uncertain descriptor system (1) is robustly admissible with a parameter-independent HPLF of degree $2q$, for all possible trajectories $w(t) \in \mathcal{W}$, if and only if the extended descriptor system

$$\mathcal{E} \begin{bmatrix} \frac{d}{dt} x^{[q]}(t) \\ \frac{d}{dt} (x^{[q-1]}(t) \otimes \xi(t)) \end{bmatrix} = \mathcal{A}_{\{q\}}(w(t)) \begin{bmatrix} x^{[q]}(t) \\ x^{[q-1]}(t) \otimes \xi(t) \end{bmatrix} \quad (14)$$

is robustly admissible with a quadratic LF, for all all possible trajectories $w(t) \in \mathcal{W}$. The definitions of $\mathcal{E}$ and $\mathcal{A}_{\{q\}}$ are given in the following, the time dependence being omitted for compactness:

$$\mathcal{E} = B \text{diag}\{I_{d_q}, 0_{d_{q-1}n_\xi}\},$$

$$\mathcal{A}_{\{q\}}(w) =$$

$$\left[\begin{array}{c|cccc} A_{11|q}(w) & M_1 A_{12}(w) & M_2 A_{12}(w) & \dots & M_{d_{q-1}} A_{12}(w) \\ \hline A_{21}(w)N_1 & A_{22}(w) & & & \\ A_{21}(w)N_2 & & A_{22}(w) & & \\ \vdots & & & \ddots & \\ A_{21}(w)N_{d_{q-1}} & & & & A_{22}(w) \end{array} \right]$$

Proof: Based on Definition III.1, the descriptor system (1) is robustly admissible with a parameter-independent HPLF of degree $2q$, for all possible trajectories $w(t) \in \mathcal{W}$, if and only if $A_{22}(w(t))$ is invertible, for all possible values of $w(t) \in \mathcal{W}$, and system (13) is robustly stable based on a parameter-independent HPLF of degree $2q$, for all possible parameters trajectories of $w(t) \in \mathcal{W}$.

The extended system obtained by applying a power transformation of degree q on the state vector x of system (13)

is expressed, according to Definition II.4, as:

$$\frac{d}{dt} x^{[q]}(t) = \frac{\partial x^{[q]}}{\partial x(t)} \Gamma(w(t))x = \Gamma_{[q]}(w(t))x^{[q]}(t). \quad (15)$$

Note that the derivative of HPLF (10) of degree $2q$ along the trajectories of system (13) equals its derivative along the trajectories of system (15) and that HPLF (10) expresses a quadratic LF regarding system (15). It follows that the descriptor system (1) is robustly admissible with a parameter-independent HPLF of degree $2q$, for all possible trajectories $w(t) \in \mathcal{W}$, if and only if $A_{22}(w(t))$ is invertible, for all possible values of $w(t) \in \mathcal{W}$, and the extended system (15) is quadratically stable for all possible parameters trajectories of $w(t) \in \mathcal{W}$. Using Lemma II.1, there exist matrices M_i , N_i , for $i = 1, \dots, d_{q-1}$, such that state matrix $\Gamma_{[q]}(w(t))$ in (15) can be rewritten as

$$\begin{aligned} \Gamma_{[q]}(w(t)) &= \sum_{i=1}^{d_{q-1}} M_i \Gamma(w(t)) N_i = \sum_{i=1}^{d_{q-1}} M_i A_{11}(w(t)) N_i \\ &\quad - \sum_{i=1}^{d_{q-1}} M_i A_{12}(w(t)) A_{22}^{-1}(w(t)) A_{21}(w(t)) N_i. \end{aligned} \quad (16)$$

The expression $\sum_{i=1}^{d_{q-1}} M_i A_{11}(w(t)) N_i$ represents the extended matrix $A_{11|q}(w(t))$ through Lemma II.1.

Introducing the notation $\mathcal{A}_{\{q\}}(w) = \begin{bmatrix} \mathcal{A}_{\{q\}11} & \mathcal{A}_{\{q\}12} \\ \mathcal{A}_{\{q\}21} & \mathcal{A}_{\{q\}22} \end{bmatrix}$ corresponding to the block-partitioning indicated in the statement of our theorem, descriptor system (14) is robustly admissible, for all possible trajectories $w(t) \in \mathcal{W}$, with a quadratic LF if and only if $\mathcal{A}_{\{q\}22}$ is invertible which is equivalent to the invertibility of $A_{22}(w(t))$, for all possible values of $w(t) \in \mathcal{W}$, and the following system

$$\frac{d}{dt} x^{[q]}(t) = \left(\mathcal{A}_{\{q\}11} - \mathcal{A}_{\{q\}12} \mathcal{A}_{\{q\}22}^{-1} \mathcal{A}_{\{q\}21} \right) x^{[q]}(t)$$

is quadratically stable. Note that this last dynamic equation is identical to the one of system (15) with description (16). This concludes the proof. ■

Based on the previous theorem, we are able to state that, when it comes to descriptor systems, the power transformation of degree q acts in a different way on its algebraic state ξ than on its dynamic state x . Indeed, the extended descriptor system is given by (14) and the associated state vector is $\begin{bmatrix} x^{[q]} \\ x^{[q-1]} \otimes \xi \end{bmatrix}$. This extended state vector is homogeneous of degree q but it does not involve all the possible monomials of degree q of the vector $\begin{bmatrix} x \\ \xi \end{bmatrix}$. The upper part of this extended state vector, $x^{[q]}$, represents the power transformation of degree q of the dynamic state x of system (1) while the lower part, $x^{[q-1]} \otimes \xi$, represents the algebraic state of the extended descriptor system. This algebraic part of the extended descriptor system is homogeneous of degree q and involves products between all algebraic state variables ξ of the original system and all the monomials of degree $q-1$ of the dynamic state x .

Remark III.1: The extended descriptor system (14) conserves the same parameter dependence as the original system (1). This is due to the fact that the block matrices composing the extended state matrix $\mathcal{A}_{\{q\}}(w)$ are linearly dependent on the matrices $A_{ij}(w)$, $i = 1, 2$, $j = 1, 2$. Since matrices $A_{ij}(w)$ are, by definition, polytopic parameter-dependent, it follows that $\mathcal{A}_{\{q\}}(w)$ is also polytopic parameter-dependent. The vertices of the polytopic parameter-dependent matrix $\mathcal{A}_{\{q\}}(w)$ are denoted as $\mathcal{A}_{i\{q\}}$, $i = 1, \dots, m$ in the following.

Based on Lemma III.1, the quadratic Lyapunov function associated to the extended descriptor system (14) and therefore, the HPLF of degree $2q$ associated to the descriptor system (1) has the general form

$$V_{2q} = \begin{bmatrix} x^{[q]}(t) \\ x^{[q-1]}(t) \otimes \xi(t) \end{bmatrix}^T \mathcal{E}^T \bar{P} \mathcal{E} \begin{bmatrix} x^{[q]}(t) \\ x^{[q-1]}(t) \otimes \xi(t) \end{bmatrix}. \quad (17)$$

Lemma III.2: The uncertain descriptor system (1) is robustly admissible with a parameter-independent HPLF of degree $2q$ for all possible trajectories of $w(t) \in \mathcal{W}$ if there exists a lower-triangular matrix $P = \begin{bmatrix} P_{11} & 0_{d_q \times d_{q-1} \cdot n_\xi} \\ P_{21} & P_{22} \end{bmatrix}$ and $\Gamma(\gamma) \in \mathcal{P}$, defined by (11), such that $P_{11} > 0$ and

$$\mathcal{A}_{\{q\}}(w(t))^T P + P \mathcal{A}_{\{q\}}(w(t)) + B \text{diag}\{\Gamma(\gamma), 0_{d_{q-1} n_\xi}\} < 0$$

for all possible trajectories of $w(t) \in \mathcal{W}$.

Proof: The LMI conditions of this theorem derive from the strict LMI admissibility conditions proposed by [22]. In fact, [22] demonstrated that a descriptor system $E\dot{X} = AX$ is admissible if and only if there exist a positive-definite matrix $\bar{P}$ and a matrix Q such that $\text{He}\{A^T(\bar{P}E + E_0Q)\} < 0$ where E_0 is the orthogonal complement matrix of E i.e. such that $E^T E_0 = 0$. Note that the Lyapunov function is $V(X) = X^T E^T \bar{P} E X$. When the system is in the SVD form, $E = B \text{diag}\{I_n, 0_{n_\xi}\}$ and then $E_0 = \begin{bmatrix} 0_{n_\xi \times n} & I_{n_\xi} \end{bmatrix}^T$. By using the notation $P = \bar{P}E + E_0Q$, it is straightforward to conclude that the above necessary and sufficient condition is equivalent to the existence of a block-partitioned matrix $P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix}$ such that $P_{11} > 0$ and $A^T P + P A < 0$. The associated Lyapunov matrix is $V(X) = X^T E^T P X$.

This result can easily be generalized to the quadratic admissibility analysis of polytopic parameter-dependent systems. Therefore, the application of this generalization to the polytopic parameter-dependent extended descriptor system (14) leads to the LMI admissibility conditions of our theorem with $\gamma_{ij} = 0$. Considering the block structure of the matrix P and the definition of derivative matrix of descriptor system (14), it follows that the Lyapunov function is $V_{2q} = x^{[q]T}(t) P_{11} x^{[q]}(t)$. The complete parameterization $P_0(\gamma)$ of the linear space (11) associated to P_{11} may be added in order to introduce complementary degrees of freedom leading to less conservative results for the admissibility analysis problem. The degrees of freedom associated to the CSMR are employed when $\gamma_{ij} \neq 0$. ■

Using the convexity principle and the notations introduced in Remark III.1, this last theorem leads to the following easily numerically tractable conditions.

Theorem III.3: The polytopic parameter-dependent descriptor system (1) is robustly admissible with a parameter-independent HPLFs of degree $2q$ for all possible trajectories of $w(t) \in \mathcal{W}$ if there exist a lower-triangular matrix $P = \begin{bmatrix} P_{11} & 0_{d_q \times d_{q-1} \cdot n_\xi} \\ P_{21} & P_{22} \end{bmatrix}$, where $P_{11} \in \mathbb{R}^{d_q \times d_q}$, $P_{21} \in \mathbb{R}^{d_{q-1} \cdot n_\xi \times d_q}$ and $P_{22} \in \mathbb{R}^{d_{q-1} \cdot n_\xi \times d_{q-1} \cdot n_\xi}$, and scalars γ_{ij} , $i = 1, \dots, N$, $j = 1, \dots, d_\mathcal{P}$, satisfying the following LMI conditions:

$$\begin{cases} P_{11} > 0 \\ \mathcal{A}_{i\{q\}}^T P + P \mathcal{A}_{i\{q\}} + \sum_{j=1}^{d_\mathcal{P}} \gamma_{ij} \cdot B \text{diag}\{P_{0j}, 0_{d_{q-1} \cdot n_\xi \times d_{q-1} \cdot n_\xi}\} < 0 \\ \text{for } i = 1, \dots, m \end{cases}$$

where matrices P_{0j} are a base of the linear space (11) (cf. Definition II.5).

Remark III.2: Note that the triangular structure of the Lyapunov matrix P employed by Lemma III.2 and Theorem III.3 is not restrictive. Indeed, as shown in the proof of Lemma III.2, the matrix P is obtained as $\bar{P}\mathcal{E} + \mathcal{E}_0\mathcal{Q}$ where $\bar{P}$ is the Lyapunov matrix in (17), $\mathcal{E}$ is the derivative matrix of the extended descriptor system (14), $\mathcal{E}_0$ is the orthogonal complement matrix of $\mathcal{E}$ and $\mathcal{Q}$ is an arbitrary matrix in $\mathbb{R}^{d_{q-1} \cdot n_\xi \times d_{q-1} \cdot n_\xi}$. Therefore, the triangular matrix P includes the degree of freedom related to the use of orthogonal complement of the derivative matrix.

The representation (17) of the Lyapunov function is not unique and its complete square matricial representation (CSMR) is given by

$$V_{2q} = \begin{bmatrix} x^{[q]}(t) \\ x^{[q-1]}(t) \otimes \xi(t) \end{bmatrix}^T \cdot \left(\mathcal{E}^T P + B \text{diag}\{\Gamma(\gamma), 0_{d_{q-1} n_\xi}\} \right) \cdot \begin{bmatrix} x^{[q]}(t) \\ x^{[q-1]}(t) \otimes \xi(t) \end{bmatrix} \quad (18)$$

where matrix $\Gamma(\gamma)$ belongs to the linear space $\mathcal{P}$ defined by (11) in Definition II.5.

B. Admissibility analysis with affine parameter-dependent HPLFs

The previous section presented robust admissibility analysis results for polytopic parameter-dependent systems descriptor systems subject arbitrarily fast parameters. However, for many parameter-dependent physical systems, there exists some information about the parameters' velocities. In the following, we elaborate robust admissibility conditions that exploit the information available on the parameters variation rate, expressed as the polyhedral convex set $\mathcal{V}$ in (3), in order to provide less conservative results. We exploit two different convexification techniques to obtain numerically tractable conditions.

Definition III.2 (Polytopic parameter-dependent homogeneous polynomial Lyapunov functions - PPD-HPLFs): A

PPD-HPLF of degree $2q$ for the uncertain descriptor system (1) is given by:

$$V_{2q} = \begin{bmatrix} x^{[q]}(t) \\ x^{[q-1]}(t) \otimes \xi(t) \end{bmatrix}^T \cdot \mathcal{E}^T P(w(t)) \mathcal{E} \cdot \begin{bmatrix} x^{[q]}(t) \\ x^{[q-1]}(t) \otimes \xi(t) \end{bmatrix}, \quad (19)$$

where $P(w(t)) \in \mathbb{R}^{(d_q+d_{q-1} \cdot n_\xi) \times (d_q+d_{q-1} \cdot n_\xi)}$ and

$$P(w(t)) = \sum_{i=1}^m w_i(t) P_i, \quad w(t) \in \mathcal{W} \text{ and } \dot{w}(t) \in \mathcal{V}. \quad (20)$$

Remark III.3: From the polytopic definition of $P(w(t))$ and since $\dot{w}(t) \in \mathcal{V}$ given in (3)-(4), it follows that:

$$\begin{aligned} P(\dot{w}(t)) &= \sum_{i=1}^m \dot{w}_i(t) P_i = \sum_{i=1}^m \sum_{j=1}^h \sigma_j(t) d_i^{(j)} P_i \\ &= \sum_{j=1}^h \sigma_j(t) \underbrace{\sum_{i=1}^m d_i^{(j)} P_i}_{=P(d^{(j)})} = \sum_{j=1}^h \sigma_j(t) P(d^{(j)}). \end{aligned} \quad (21)$$

Definition III.3 (CSMR of PPD-HPLFs for descriptor systems): The CSMR associated to (19) is given by

$$\begin{aligned} V_{2q} &= \begin{bmatrix} x^{[q]}(t) \\ x^{[q-1]}(t) \otimes \xi(t) \end{bmatrix}^T \cdot (\mathcal{E}^T P(w(t)) \mathcal{E} \\ &+ B \text{diag}\{\Gamma(\gamma), 0_{d_{q-1}n_\xi}\}) \cdot \begin{bmatrix} x^{[q]}(t) \\ x^{[q-1]}(t) \otimes \xi(t) \end{bmatrix}. \end{aligned} \quad (22)$$

As stated in Definition II.5, matrix $\Gamma(\gamma)$ is the span of the set $\mathcal{P}$ defined by (11).

Remark III.4: In the work [10, Theorem 1], some necessary and sufficient conditions for the admissibility analysis of linear time-varying (LTV) descriptor systems have been presented. Note that these conditions are necessary and sufficient only for the existence of a quadratic Lyapunov function. In the present paper, we assess the admissibility of a parameter-dependent system by providing necessary and sufficient conditions for the existence of Lyapunov functions belonging to a larger class, namely the class of homogeneous polynomial Lyapunov functions. This class of Lyapunov functions is universal for uncertain systems, and its use might lead to less conservative robust admissibility results.

Our next lemma extends the results presented in Lemma III.2 to PPD-HPLFs by exploiting the information available on the parameters variation rate. In this lemma, the degree of freedom related to the orthogonal complement of the derivative matrix $\mathcal{E}$ appears explicitly in the formulation of the inequality conditions while, in Lemma III.2, this degree of freedom is hidden in the (2,2)-block of the Lyapunov matrix (see Remark III.2).

Lemma III.4: The uncertain descriptor system (1) subject to time-varying parameters with bounded rates of variation is robustly admissible with a PPD-HPLF of degree $2q$ if and only if one of the following equivalent sets of conditions is satisfied:

(i) there exist matrices $P(w(t)), Q(w(t)) \in \mathbb{R}^{(d_{q-1} \cdot n_\xi) \times (d_q+d_{q-1} \cdot n_\xi)}$ and $\Gamma(\gamma(w, \dot{w}))$ such that

$$\begin{cases} P(w(t)) > 0 \\ \text{He}\{\mathcal{A}_{\{q\}}^T(w(t))(P(w(t))\mathcal{E} + \mathcal{E}_0 Q(w(t)))\} \\ + \mathcal{E}^T P(\dot{w}(t))\mathcal{E} + B \text{diag}\{\Gamma(\gamma(w, \dot{w})), 0_{d_{q-1} \cdot n_\xi}\} < 0 \end{cases} \quad (23)$$

for all $w(t) \in \mathcal{W}$ and $\dot{w}(t) \in \mathcal{V}$.

(ii) there exist matrices $P(w(t)), F(w(t))$ and $G(w(t)) \in \mathbb{R}^{(d_q+d_{q-1} \cdot n_\xi) \times (d_q+d_{q-1} \cdot n_\xi)}$, $Q(w(t)) \in \mathbb{R}^{(d_{q-1} \cdot n_\xi) \times (d_q+d_{q-1} \cdot n_\xi)}$, and $\Gamma(\gamma(w, \dot{w}))$ such that

$$\begin{cases} P(w(t)) > 0 \\ \left[\begin{array}{c|c} \mathcal{E}^T P(\dot{w}(t))\mathcal{E} + \text{He}\{F(w(t))\mathcal{A}_{\{q\}}(w(t))\} + B \text{diag}\{\Gamma(\gamma(w, \dot{w})), 0_{d_{q-1} \cdot n_\xi}\} & \star \\ \hline P(w(t))\mathcal{E} + \mathcal{E}_0 Q(w(t)) + G(w(t))\mathcal{A}_{\{q\}}(w(t)) - F^T(w(t)) & -\text{He}\{G(w(t))\} \end{array} \right] < 0 \end{cases} \quad (24)$$

for all $w(t) \in \mathcal{W}$ and $\dot{w}(t) \in \mathcal{V}$.

Matrix $\mathcal{E}_0 \in \mathbb{R}^{(d_q+d_{q-1} \cdot n_\xi) \times (d_{q-1} \cdot n_\xi)}$ is the orthogonal complement of $\mathcal{E}$ while $\mathcal{A}_{\{q\}}(w(t))$ is defined in Lemma III.1. The Lyapunov function associated to the above sets of conditions is given by (19).

Proof: (i): Following the lines of the proof of Lemma III.1, the admissibility of a descriptor system by means of a HPLF of degree $2q$ and of a given parameter dependence is equivalent to the admissibility of its associated extended system by means of a LF that is quadratic in its extended state vector and that presents the same parameter dependence of the initial HPLF. This is due to the fact the LFs certifying the admissibility of both systems are identical. This means that the uncertain descriptor system (1) is admissible with a PPD-HPLF of degree $2q$ if and only if its extended system is admissible with a polytopic parameter-dependent LF that is quadratic on the states of the extended system.

Therefore, we look for a polytopic parameter-dependent LF candidate that is positive-definite and whose time derivative along the trajectories of the extended system (14) is negative-definite. Positive-definiteness of LF is verified in the first inequality of (i), while the second inequality of (i) results from the negativity condition of the LF time-derivative and the equality

$$\begin{bmatrix} x^{[q]}(t) \\ x^{[q-1]}(t) \otimes \xi(t) \end{bmatrix}^T \cdot B \text{diag}\{\Gamma(\gamma(w, \dot{w})), 0_{d_{q-1} \cdot n_\xi}\} \cdot \begin{bmatrix} x^{[q]}(t) \\ x^{[q-1]}(t) \otimes \xi(t) \end{bmatrix} = 0.$$

Equivalences: (i) $\rightarrow$ (ii): Consider the second strict inequality of (23). Then, there exists a small scalar ε such that

$$\text{left side of inequality (23)} < -\frac{\varepsilon}{2} \mathcal{A}_{\{q\}}^T(w(t)) \mathcal{A}_{\{q\}}(w(t)).$$

This last inequality is equivalent, by Schur complement, to the second inequality in (24) when F and G are chosen as $F = \mathcal{E}^T P(w(t)) + Q^T(w(t)) \mathcal{E}_0^T$ and $G = \varepsilon \cdot I_{d_q+d_{q-1} \cdot n_\xi}$.

(ii) $\rightarrow$ (i): Left- and right-multiplying the second inequality in (24) by $\begin{bmatrix} I_{d_q+d_{q-1} \cdot n_\xi} & \mathcal{A}_{\{q\}}^T(w(t)) \end{bmatrix}$ and its transpose, respectively, leads to the second inequality of (23). ■

Remark III.5: Some complementary degrees of freedom associated to the conditions (i) and (ii) of Theorem III.4 can be highlighted. The first of them is related to the CMSR described in Definition II.5, caused by the power transformation of the state vectors. The second of them is related to the rank-deficiency of the matrix $\mathcal{E}$, and it can be exploited by the orthogonal complement of this matrix. The orthogonal complement has also been used in [10, Theorem 1].

Remark III.6: The negative-definiteness of the derivative of the Lyapunov function can accommodate the so-called slack variables, that appear through the use of the projection lemma and are referred as $F(w(t))$ and $G(w(t))$ in (24). When the infinite amount of conditions to be observed (*i.e.* inequality (24) must be verified for every possible values of parameters and parameter's rate of variation) is brought to a finite amount of conditions due to a certain choice on the structure of the unknown matrices, the slack variables might provide extra degrees of freedom, leading to less conservative results.

The admissibility conditions presented in Lemma III.4 might lead to nonlinear matrix inequalities. In the following, two different structures of the involved slack variables are chosen in order to obtain new less restrictive LMI admissibility conditions for continuous-time uncertain descriptor systems subject to parameters with bounded rates of variation. The first structure consists of constant slack variables and is presented in the theorem below.

Theorem III.5: The parameter-dependent uncertain descriptor system (1) subject to parameters with bounded rates of variation is robustly admissible based on a PPD-HPLF of degree $2q$ if there exist matrices $P_i > 0$, Q_i , for $i = 1 \dots m$, F and G of the same dimensions as in Lemma III.4, and a set of vectors $\gamma^{(il)} \in \mathbb{R}^{d_{\mathcal{E}}}$, $i = 1 \dots m$, $l = 1 \dots h$, satisfying the LMI conditions below:

$$\left[\begin{array}{c|c} \mathcal{E}^T P(d^{(l)}) \mathcal{E} + \text{He}\{F \mathcal{A}_{i\{q\}}\} & \star \\ + B \text{diag}\{\Gamma(\gamma^{(il)}), 0_{d_{q-1} n_{\mathcal{E}}}\} & \\ \hline P_i \mathcal{E} + \mathcal{E}_0 Q_i + G \mathcal{A}_{i\{q\}} - F^T & -\text{He}\{G\} \end{array} \right] < 0, \quad (25)$$

for $i = 1, \dots, m$ and $l = 1, \dots, h$.

Matrices $\mathcal{A}_{i\{q\}}$ are the vertices of $\mathcal{A}_{\{q\}}(w)$, as referred in Remark III.1.

Proof: Condition (25) is directly derived from (24) through the particular choice of constant slack variables $F(w(t)) = F$ and $G(w(t)) = G$, and by considering polytopic realizations of matrices $P(w(t))$ (20), $P(\dot{w}(t))$ (21), and $Q(w(t))$:

$$Q(w(t)) = \sum_{i=1}^m w_i(t) Q_i. \quad (26)$$

We recall that the term $\Gamma(\gamma(w, \dot{w}))$ in (24) spans a linear space, and that the dependence of $\gamma(w, \dot{w})$ with respect to the parameters and parameter's velocities can be conveniently chosen in such a way that the term $\Gamma(\gamma(w, \dot{w}))$ in

(25) is convex with respect to these both independent sets:

$$\Gamma(\gamma(w, \dot{w})) = \sum_{i=1}^m \sum_{l=1}^h w_i \sigma_l \Gamma(\gamma^{(il)}). \quad \blacksquare$$

Remark III.7: Theorem III.5 covers the results provided in [10] that correspond to the particular case $q = 1$,

Our next theorem considers parameter-dependent slack variables.

Theorem III.6: The uncertain descriptor system (1) subject to parameters with bounded rates of variation is robustly admissible based on a PPD-HPLF of degree $2q$ if there exist matrices $P_i > 0$, F_i , G_i and Q_i , $i = 1, \dots, m$, and sets of vectors $\phi^{(il)} \in \mathbb{R}^{d_{\mathcal{E}}}$, for $i = 1, \dots, m$, $l = 1, \dots, h$, $\Delta^{(ijl)} \in \mathbb{R}^{d_{\mathcal{E}}}$, for $i, j = 1, \dots, m$, $j \neq i$, $l = 1, \dots, h$, $\Omega^{(ijkl)} \in \mathbb{R}^{d_{\mathcal{E}}}$, for $i = 1, \dots, m-2$, $j = i+1, \dots, m-1$, $k = j+1, \dots, m$, $l = 1, \dots, h$, such that the LMI conditions (27) are satisfied.

Proof: Consider the polytopic definition of the system (1) and of matrices $P(w(t))$ in (20), $P(\dot{w}(t))$ in (21), $Q(w(t))$ in (26), and $F(w(t))$, $G(w(t))$ respectively as $F(w(t)) = \sum_{i=1}^m w_i(t) F_i$ and $G(w(t)) = \sum_{i=1}^m w_i(t) G_i$ for $w(t) \in \mathcal{W}$ and $\dot{w}(t) \in \mathcal{V}$.

Using Lemma III.4, system (1) is robustly admissible if and only if (24) holds. Once we replace the polytopic definitions of the matrices above in (24), we obtain a matrix inequality composed by terms with polynomial dependence on w of degrees 0, 1 or 2, and of polynomial dependence on σ of degrees 0 or 1. Inspired by the technique employed in [33], we cast every term of the matrix inequality into a form with polynomial dependence of degree 3 in w and of degree 1 in σ . This is obtained multiplying the term

$$\begin{aligned} \sum_{l=1}^h \sigma_l \mathcal{E}^T P(d^{(l)}) \mathcal{E} & \quad \text{by} \quad \left(\sum_{i=1}^m w_i \right)^3, \\ \text{He}\left\{ \sum_{i=1}^m w_i F_i \sum_{i=1}^m w_i \mathcal{A}_{i\{q\}} \right\} & \quad \text{by} \quad \left(\sum_{i=1}^m w_i \right) \left(\sum_{l=1}^h \sigma_l \right), \\ \sum_{i=1}^m w_i P_i \mathcal{E} & \quad \text{by} \quad \left(\sum_{i=1}^m w_i \right)^2 \left(\sum_{l=1}^h \sigma_l \right), \\ \sum_{i=1}^m w_i \mathcal{E}_0 Q_i & \quad \text{by} \quad \left(\sum_{i=1}^m w_i \right)^2 \left(\sum_{l=1}^h \sigma_l \right), \\ \sum_{i=1}^m w_i G_i \sum_{i=1}^m w_i \mathcal{A}_{i\{q\}} & \quad \text{by} \quad \left(\sum_{i=1}^m w_i \right) \left(\sum_{l=1}^h \sigma_l \right), \\ - \sum_{i=1}^m w_i F_i^T & \quad \text{by} \quad \left(\sum_{i=1}^m w_i \right)^2 \left(\sum_{l=1}^h \sigma_l \right), \\ - \sum_{i=1}^m w_i \text{He}\{G_i\} & \quad \text{by} \quad \left(\sum_{i=1}^m w_i \right)^2 \left(\sum_{l=1}^h \sigma_l \right) \end{aligned}$$

since w and σ are polytopic variables satisfying $\sum_{i=1}^m w_i = 1$

and $\sum_{l=1}^h \sigma_l = 1$.

Note that, in the structure of the PPD-HPLF in (22), the matrix $\Gamma(\gamma)$ has no predefined dependence on the parameter vector w . Therefore, we can choose a convenient parameteri-

$$\left[\begin{array}{c|c} \mathcal{E}^T P(d^{(l)}) \mathcal{E} + \text{He}\{F_i \mathcal{A}_{i\{q\}}\} + B \text{diag}\{\Gamma(\phi^{(il)}), 0_{d_{q-1}, n_\xi}\} & \star \\ \hline P_i \mathcal{E} + \mathcal{E}_0 Q_i + G_i \mathcal{A}_{i\{q\}} - F_i^T & -\text{He}\{G_i\} \end{array} \right] < - \begin{bmatrix} I & \\ & 0 \end{bmatrix},$$

for $i = 1, \dots, m, l = 1, \dots, h$.

$$\left[\begin{array}{c|c} 3\mathcal{E}^T P(d^{(l)}) \mathcal{E} + \text{He}\{F_i \mathcal{A}_{j\{q\}} + F_j \mathcal{A}_{i\{q\}} + F_i \mathcal{A}_{i\{q\}}\} + B \text{diag}\{\Gamma(\Delta^{(ijl)}), 0_{d_{q-1}, n_\xi}\} & \star \\ \hline 2(P_i \mathcal{E} + \mathcal{E}_0 Q_i - F_i^T) + (P_j \mathcal{E} + \mathcal{E}_0 Q_j - F_j^T) + G_i \mathcal{A}_{j\{q\}} + G_j \mathcal{A}_{i\{q\}} + G_i \mathcal{A}_{i\{q\}} & -\text{He}\{2G_i + G_j\} \end{array} \right] < \frac{1}{(m-1)^2} \begin{bmatrix} I & \\ & 0 \end{bmatrix},$$

for $i = 1, \dots, m, j = 1, \dots, m, j \neq i, l = 1, \dots, h$. (27)

$$\left[\begin{array}{c|c} 6\mathcal{E}^T P(d^{(l)}) \mathcal{E} + \text{He}\{F_i \mathcal{A}_{j\{q\}} + F_i \mathcal{A}_{k\{q\}} + F_j \mathcal{A}_{k\{q\}} + F_j \mathcal{A}_{i\{q\}} + F_k \mathcal{A}_{i\{q\}} + F_k \mathcal{A}_{j\{q\}}\} + B \text{diag}\{\Gamma(\Omega^{(ijkl)}), 0_{d_{q-1}, n_\xi}\} & \star \\ \hline \frac{2(P_i + P_j + P_k) \mathcal{E} + 2\mathcal{E}_0(Q_i + Q_j + Q_k) - 2(F_i^T + F_j^T + F_k^T)}{+ G_i \mathcal{A}_{j\{q\}} + G_j \mathcal{A}_{i\{q\}} + G_i \mathcal{A}_{k\{q\}} + G_k \mathcal{A}_{i\{q\}} + G_j \mathcal{A}_{k\{q\}} + G_k \mathcal{A}_{j\{q\}}} & -2\text{He}\{G_i + G_j + G_k\} \end{array} \right] < \frac{6}{(m-1)^2} \begin{bmatrix} I & \\ & 0 \end{bmatrix},$$

for $i = 1, \dots, m-2, j = i+1, \dots, m-1, k = j+1, \dots, m, l = 1, \dots, h$.

zation of γ in order to provide one independent realization of $\mathcal{P}$ for every LMI condition to be generated from inequality (24). The following choice has been made

$$\begin{aligned} \Gamma(\gamma(w, \dot{w})) &= \sum_{l=1}^h \sum_{i=1}^m \sigma_l w_i^3 \Gamma(\phi^{(il)}) \\ &+ \sum_{l=1}^h \sum_{i=1}^m \sum_{j=1, j \neq i}^m \sigma_l w_i^2 w_j \Gamma(\Delta^{(ijl)}) \\ &+ \sum_{l=1}^h \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^m \sigma_l w_i w_j w_k \Gamma(\Omega^{(ijkl)}). \end{aligned}$$

With the recasting of each term and the defined structure of γ , the left side of (24) is given as in (28).

Whenever inequalities (27) hold, it follows that the left hand side of (24), expressed as in (28), satisfies inequality (29). Since the scalar

$$\begin{aligned} &\sum_{l=1}^h \sigma_l \left(\sum_{i=1}^m w_i^3 - \frac{1}{(m-1)^2} \sum_{\substack{j=1; \\ j \neq i}}^m w_i^2 w_j \right. \\ &\quad \left. - \frac{6}{(m-1)^2} \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^m w_i w_j w_k \right) \end{aligned}$$

can be rewritten as:

$$\begin{aligned} &\frac{1}{(m-1)^2} \sum_{l=1}^h \sigma_l \left((m-1)^2 \sum_{i=1}^m w_i^3 - \sum_{\substack{j=1; \\ j \neq i}}^m w_i^2 w_j \right. \\ &\quad \left. - 6 \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^m w_i w_j w_k \right) = \frac{1}{(m-1)^2} \sum_{l=1}^h \sigma_l \left((m-1) \Theta + \frac{\Omega}{2} \right), \end{aligned}$$

where $\Theta = \sum_{i=1}^m \sum_{j=1}^m w_i (w_i - w_j)^2$ and $\Omega = \sum_{i=1}^m \sum_{j=1; k=1; j \neq i}^m \sum_{k=j+1}^m w_k (w_i - w_j)^2$,

it is positive semidefinite.

Therefore, the left hand side of (24) is negative-definite *i.e.* (24) is satisfied, and the system (1) is admissible. ■

Remark III.8: Theorem III.6 also covers results proposed in the literature. When using a quadratic Lyapunov function (*i.e.* $q = 1$) and disconsidering the degrees of freedom associated

to the CSMR (*i.e.* $\gamma = 0$) for the stability analysis of a standard continuous-time uncertain system (*e.g.* $n_\xi = 0, E = I_n$), our conditions cover the ones found in [33].

The ensuing section brings numerical results that show reduction in the conservatism when using the proposed LMI conditions.

IV. NUMERICAL EXAMPLES

In this section, numerical examples are proposed to evaluate the performance of the LMI conditions of Theorem III.3, Theorem III.5 and Theorem III.6. First, we show that parameter-independent HPLFs can provide, through the LMI conditions of Theorem III.3, less conservative admissibility analysis results for uncertain descriptor systems with arbitrary parameters variation rate. Then, we compare the admissibility analysis LMI conditions of Theorem III.5 and Theorem III.6 with results available in the literature for uncertain descriptor systems subject to different bounds on parameters' velocities. In addition, we evaluate the influence of the degree of PPD-HPLFs on the performance of our new LMI results.

A. Admissibility analysis using parameter-independent HPLFs

Consider the descriptor system defined by state matrices $E = B \text{diag}\{I_3, 0\}$ and

$$A(\alpha(t)) = \begin{bmatrix} -2\alpha(t) & 1 & -\alpha(t) & 0.5 \\ \alpha(t) & -10\alpha(t) & 1+3\alpha(t) & 1+\alpha(t) \\ -1+\alpha(t) & -2-4\alpha(t) & -4+2\alpha(t) & 0 \\ 1-\alpha(t) & 0 & 0 & 1+0.2\alpha(t) \end{bmatrix}$$

where the parameter $\alpha(t)$ is bounded, $0 \leq \alpha(t) \leq \kappa$, and has an arbitrary variation rate. The number of dynamic and algebraic states is $n = 3$ and, respectively, $n_\xi = 1$. This system can be rewritten in the polytopic parameter-dependent form (1)-(2) with $m = 2$ parameters, $w_1(t) = \alpha(t)/\kappa$ and

$$\begin{aligned}
\text{left-hand side of (24)} &= \sum_{l=1}^h \sigma_l \sum_{i=1}^m w_i^3 \left[\frac{\mathcal{E}^T P(d^{(l)}) \mathcal{E} + \text{He}\{F_i \mathcal{A}_{i\{q\}}\} + \text{Bdiag}\{\Gamma(\phi^{(il)}), 0_{d_{q-1} \cdot n_\xi}\}}{P_i \mathcal{E} + \mathcal{E}_0 Q_i + G_i \mathcal{A}_{i\{q\}} - F_i^T} \mid \star \right] \\
&+ \sum_{l=1}^h \sigma_l \sum_{i=1}^m \sum_{j=1, j \neq i}^m w_i^2 w_j \left[\frac{3 \mathcal{E}^T P(d^{(l)}) \mathcal{E} + \text{He}\{F_i \mathcal{A}_{j\{q\}} + F_j \mathcal{A}_{i\{q\}} + F_i \mathcal{A}_{i\{q\}}\} + \text{Bdiag}\{\Gamma(\Delta^{(ijl)}), 0_{d_{q-1} \cdot n_\xi}\}}{2(P_i \mathcal{E} + \mathcal{E}_0 Q_i - F_i^T) + (P_j \mathcal{E} + \mathcal{E}_0 Q_j - F_j^T) + G_i \mathcal{A}_{j\{q\}} + G_j \mathcal{A}_{i\{q\}} + G_i \mathcal{A}_{i\{q\}}} \mid \star \right] \\
&+ \sum_{l=1}^h \sigma_l \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^m w_i w_j w_k \left[\frac{6 \mathcal{E}^T P(d^{(l)}) \mathcal{E} + \text{He}\{F_i \mathcal{A}_{j\{q\}} + F_i \mathcal{A}_{k\{q\}} + F_j \mathcal{A}_{k\{q\}} + F_j \mathcal{A}_{i\{q\}} \right. \\
&\quad \left. + F_k \mathcal{A}_{i\{q\}} + F_k \mathcal{A}_{j\{q\}}\} + \text{Bdiag}\{\Gamma(\Omega^{(ijkl)}), 0_{d_{q-1} \cdot n_\xi}\}}{2(P_i + P_j + P_k) \mathcal{E} + 2 \mathcal{E}_0 (Q_i + Q_j + Q_k) - 2(F_i^T + F_j^T + F_k^T) + G_i \mathcal{A}_{j\{q\}} + G_j \mathcal{A}_{i\{q\}} + G_i \mathcal{A}_{k\{q\}} + G_k \mathcal{A}_{i\{q\}} + G_j \mathcal{A}_{k\{q\}} + G_k \mathcal{A}_{j\{q\}}} \mid \star \right] \\
&\quad \left. - 2 \text{He}\{G_i + G_j + G_k\} \right]
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\left[\frac{\mathcal{E}^T \dot{P}(w(t)) \mathcal{E} + \text{He}\{F(w(t)) \mathcal{A}_{\{q\}}(w(t))\} + \text{Bdiag}\{\Gamma(\gamma(w, \dot{w})), 0_{d_{q-1} \cdot n_\xi}\}}{P(w(t)) \mathcal{E} + \mathcal{E}_0 Q(w(t)) + G(w(t)) \mathcal{A}_{\{q\}}(w(t)) - F^T(w(t))} \mid \star \right] \\
&\quad \left[\frac{P(w(t)) \mathcal{E} + \mathcal{E}_0 Q(w(t)) + G(w(t)) \mathcal{A}_{\{q\}}(w(t)) - F^T(w(t))}{- \text{He}\{G(w(t))\}} \right] < \left(- \sum_{l=1}^h \sigma_l \left(\sum_{i=1}^m w_i^3 - \frac{1}{(m-1)^2} \sum_{j=1, j \neq i}^m w_i^2 w_j \right. \right. \\
&\quad \left. \left. - \frac{6}{(m-1)^2} \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^m w_i w_j w_k \right) \begin{bmatrix} I & 0 \end{bmatrix} \right)
\end{aligned} \tag{29}$$

$w_2(t) = 1 - \alpha(t)/\kappa$, and vertex state matrices

$$A^{(1)} = \begin{bmatrix} 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 1 \\ -1 & -2 & -4 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$A^{(2)} = \begin{bmatrix} -2\kappa & 1 & -\kappa & 0.5 \\ \kappa & -10\kappa & 1+3\kappa & 1+\kappa \\ -1+3\kappa & -2-4\kappa & -4+2\kappa & 0 \\ 1-\kappa & 0 & 0 & 1+0.2\kappa \end{bmatrix}.$$

$$[M_1 \mid M_2 \mid M_3] = \left[\begin{array}{ccc|ccc|ccc} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{array} \right],$$

$$[N_1^T \mid N_2^T \mid N_3^T] = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

Since the parameter $\alpha(t)$ variation rate is arbitrary, only parameter-independent LF can be used for admissibility analysis. Hence, the admissibility property has to be guaranteed for all possible, arbitrarily fast, trajectories of $w(t) = [w_1(t) \ w_2(t)]^T$ in $\mathcal{W}$.

The robust admissibility of this system is studied in the following using the result proposed in Theorem III.3 for different degrees of HPLFs. For the sake of illustration, in the case of HPLFs of degree 4 (i.e. $q = 2$), $x^{[2]}$ is of dimension $d_q = 6$ and equals $[x_1^2 \ x_1 x_2 \ x_1 x_3 \ x_2^2 \ x_2 x_3 \ x_3^2]^T$, the dimension of the linear space (11) is $d_{\mathcal{P}} = 6$ and the parameterization of the HPLF of degree 2 is

$$P_0(\gamma) = \begin{bmatrix} 0 & 0 & 0 & -\gamma_1 & -\gamma_2 & -\gamma_3 \\ 0 & 2\gamma_1 & \gamma_2 & 0 & -\gamma_4 & -\gamma_5 \\ 0 & \gamma_2 & 2\gamma_3 & \gamma_4 & \gamma_5 & 0 \\ -\gamma_1 & 0 & \gamma_4 & 0 & 0 & -\gamma_6 \\ -\gamma_2 & -\gamma_4 & \gamma_5 & 0 & 2\gamma_6 & 0 \\ -\gamma_3 & -\gamma_5 & 0 & -\gamma_6 & 0 & 0 \end{bmatrix}.$$

Matrices M_i and N_i , for $i = 1, \dots, d_{q-1}$ where $d_{q-1} = 3$, in Lemma II.1, are given by

When the parameter α is uncertain and constant, by using the root-locus of the finite modes of the descriptor system, we obtain the largest value of the bound κ for which the system is admissible as 16.70083. When the parameter α is time-varying with arbitrary variation rate, quadratic LFs ($q = 1$) guarantee the system admissibility for all parameter trajectories when $\kappa \leq 4.6667$. Theorem III.3 proves, based on HPLFs of degree 4 ($q = 2$) and 6 ($q = 3$), that the admissibility property is preserved when $\kappa \leq 16.3943$ and, respectively when $\kappa \leq 16.7008$. This shows a significant improvement, even for a slight augmentation of the HPLF degree, of the largest value of the bound κ for which the system is admissible. Therefore, this indicates that the HPLF approach proposed in this paper allows to obtain less conservative admissibility analysis results when a suitable polynomial degree $2q$ is chosen.

B. Admissibility analysis using parameter-dependent HPLFs

In order to provide a numerical comparison between our new results and results available in the literature, the following two systems are considered.

System IV.1: Consider $E = \text{Bdiag}\{I_2, 0\}$,

$$A(r(t)) = \begin{bmatrix} -5+2r(t) & -2+r(t) & 0.1r(t) \\ 2 & -r(t) & 0 \\ 0 & 10r(t) & 1+0.1r(t) \end{bmatrix}$$

where $0 \leq r(t) \leq \kappa$ and $|\dot{r}(t)| \leq \beta$,

recast in the polytopic parameter-dependent form (1)-(2) with $m = 2$ parameters, $w_1(t) = r(t)/\kappa$ and $w_2(t) = 1 - r(t)/\kappa$, and vertex state matrices

$$A^{(1)} = \begin{bmatrix} -5 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} -5+2\kappa & -2+\kappa & 0.1\kappa \\ 2 & -\kappa & 0 \\ 0 & 10\kappa & 1+0.1\kappa \end{bmatrix}.$$

System IV.2: Consider $E = B \text{diag}\{I_2, 0\}$,

$$A(r(t)) = \begin{bmatrix} 0 & 1 & 0.2 \\ -6-r(t) & -1 & 0 \\ 0.1 & 0.2 & 1+0.1r(t) \end{bmatrix},$$

where $-\kappa \leq r(t) \leq \kappa$ and $|\dot{r}(t)| \leq \beta$,

reformulated in the polytopic parameter-dependent form (1)-(2) with $m = 2$ parameters, $w_1(t) = r(t)/\kappa$ and $w_2(t) = 1 - r(t)/\kappa$, and vertex state matrices

$$A^{(1)} = \begin{bmatrix} 0 & 1 & 0.2 \\ -6+\kappa & -1 & 0 \\ 0.1 & 0.2 & 1-0.1\kappa \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} 0 & 1 & 0.2 \\ -6-\kappa & -1 & 0 \\ 0.1 & 0.2 & 1+0.1\kappa \end{bmatrix}.$$

For both systems, the number of dynamic and algebraic states is $n = 2$ and, respectively $n_\xi = 1$. From $|\dot{r}(t)| \leq \beta$, it follows that $|\dot{w}_1(t)| \leq \beta/\kappa$ and $|\dot{w}_2(t)| \leq \beta/\kappa$. Then, the set of all admissible parameters trajectories is given by $w(t) = [w_1(t) \ w_2(t)]^T \in \mathcal{W}$ along with $\dot{w}(t) \in \mathcal{V}$ where the polyhedral convex set $\mathcal{V}$ is defined by (3) with $h = 2$ and $\{d^{(1)}, d^{(2)}\} = \left\{ \begin{bmatrix} -\beta/\kappa \\ \beta/\kappa \end{bmatrix}, \begin{bmatrix} \beta/\kappa \\ -\beta/\kappa \end{bmatrix} \right\}$. For the sake of illustration, since $n = 2$, if we consider HPLFs of degree 4 ($q = 2$) then $d_q = 3$ and $d_{q-1} = 2$. The power transformation of degree 2 of x is $x^{[2]} = [x_1^2 \ x_1x_2 \ x_2^2]^T$ and Lemma II.1 holds with $[M_1 \mid M_2] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$,

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{ For HPLFs of degree 6 } (q = 3),$$

we have $d_q = 4$, $d_{q-1} = 3$, $x^{[3]} = [x_1^3 \ x_1^2x_2 \ x_1x_2^2 \ x_2^3]^T$ and the matrices M_i and N_i in Lemma II.1 are given by

$$\begin{bmatrix} M_1^T \\ M_2^T \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We are interested in finding, for different values of β , the largest value $\kappa_{2q}^*(\beta)$ of κ for which the uncertain parameter-dependent descriptor System IV.1 and System IV.2 are robustly admissible in the uncertainty domain based on HPLFs of degree $2q$.

1) *Evaluation of $\kappa_{2q}^*(\beta)$ in comparison with results from the literature:* The values of $\kappa_{2q}^*(\beta)$ for System IV.1 and System IV.2 have been assessed using our LMI conditions given in Theorem III.3, Theorem III.5 and Theorem III.6, and the ones presented in the work of [11], *Theorem 3.1*, and [10], *Lemma 3*. We recall that the technique provided in the paper of [10], *Lemma 3* is covered by *Theorem III.5* for the choice $q = 1$ (quadratic LF). The results of our evaluation are reported in Tables I and II.

TABLE I: $\kappa_{2q}^*(50)$ for System IV.1

	$\kappa_{2q}^*(50)$
[11], <i>Th.3.1</i>	2.4434
[10], <i>Lemma 3</i>	2.4434
Theorem III.3	q=1 2.433
	q=2 2.899
	q=3 3.107
	q=4 3.132
	q=5 3.179
	q=6 3.195
	q=7 3.206
Theorem III.5	q=1 2.4434
	q=2 2.9097
	q=3 3.1074
	q=4 3.1449
	q=5 3.1855
	q=6 3.2022
	q=7 3.213
Theorem III.6	q=1 2.4471
	q=2 2.9182
	q=3 3.1101
	q=4 3.1496
	q=5 3.1853
	q=6 3.2033
	q=7 3.2064

From Tables I and II, we notice that LMI conditions of Theorem III.5 and Theorem III.6 lead to larger values of $\kappa_{2q}^*(\beta)$ than the techniques proposed in the literature. This is due to the fact that the result proposed in [10] is covered by Theorem III.5 while the condition proposed in [11] uses less degrees of freedom related to the use of slack variables than this latter reference. Note that Theorem III.3 also leads to larger values of $\kappa_{2q}^*(\beta)$ for System IV.1 with $\beta = 50$ and for System IV.2 with $\beta = 100$.

Throughout Tables I and II, we remark that, for a fixed degree of Lyapunov function, sometimes Theorem III.5 gives better results than Theorem III.6, other times the converse holds. This is due to the fact that Theorems III.5 and III.6, as the results previously presented in the literature, are only sufficient and they might not spam the same space of solutions.

TABLE II: $\kappa_{2q}^*(\beta)$ for System IV.2

		$\kappa_{2q}^*(5)$	$\kappa_{2q}^*(100)$
[11], Th.3.1		5.0139	2.5896
[10], Lemma 3		5.0304	2.5897
Theorem III.3	q=1	2.572	
	q=2	3.451	
	q=3	3.650	
	q=4	3.715	
	q=5	3.817	
	q=6	3.820	
	q=7	3.866	
Theorem III.5	q=1	5.0304	2.5897
	q=2	5.2756	3.4525
	q=3	5.1832	3.6588
	q=4	5.0798	3.7318
	q=5	4.9715	3.8236
	q=6	4.8779	3.8482
	q=7	4.8083	3.8728
Theorem III.6	q=1	5.3228	2.5898
	q=2	5.2797	3.4531
	q=3	5.2423	3.6659
	q=4	5.1124	3.733
	q=5	4.986	3.826
	q=6	4.89	3.8401
	q=7	4.8131	3.8669

In addition, we can observe from the evaluation reported in these tables that a reduction of the conservatism (larger values of $\kappa_{2q}^*(\beta)$) can be obtained through an augmentation of the HPLF degree for System IV.1 with $\beta = 50$ as well as for System IV.2 with $\beta = 100$. However, this is not always the case, as reflected by the evaluation of System IV.2 with $\beta = 5$, since an increase then a decrease of $\kappa_{2q}^*(\beta)$ may appear through the augmentation of the HPLF degree. This point is discussed in the following evaluation.

2) *Conservativeness evaluation regarding the augmentation of the HPLFs degree:* For System IV.2, the maximum bounds β and κ for which the admissibility is guaranteed based on Theorem III.5 with PPD-HPLFs of different degrees, have been computed and plotted in Figure 1. We notice from this figure that the curves of different degrees of PPD-HPLFs overlap for this particular system. For instance, among the performed tests, the PPD-HPLF providing the biggest allowed velocity of parameters when $\kappa_{2q}^* = 4$ is the one of order 14, whilst for $\kappa_{2q}^* = 5$, the best choice of degree for a PPD-HPLF is 6.

Fig. 1: System IV.2: Maximum bound of β and κ_{2q}^* defining the uncertainty domain for which the admissibility is guaranteed based on Theorem III.5 for PPD-HPLFs of degree $2q$

As a matter of fact, the results reported in Figure 1 follow the same tendency displayed in the work of [5] for some standard uncertain systems ($E = I_n$). Indeed, in [5], it is highlighted that, for Lyapunov functions with parameter dependence of any degree higher than or equal to one, a mere augmentation on the degree of the state variables does not necessarily lead to less conservative results. That, once more, highlights one open problem concerning parameter-dependent homogeneous polynomial Lyapunov functions: there is no clear rule for choosing its best suitable degree for a given system and given parameters domain. It also shows that different PPD-HPLFs might provide different sets of necessary conditions, so that PPD-HPLFs of higher degrees do not necessarily span the same space of solutions of PPD-HPLFs of lower degrees. As a matter of fact, the authors of [5] state that, for standard uncertain systems, the joint augmentation on the degree of the homogeneous polynomial Lyapunov function with respect to the state and the parameter vectors might lead to less restrictive conditions. However, there is no guarantee that conservatism will decrease for an augmentation of the degree of the power transformation of the state vector alone. This fact has also been confirmed by our experiments in the case of descriptor uncertain systems.

V. CONCLUSION

The present paper presents new robust admissibility analysis conditions for continuous-time polytopic parameter-dependent uncertain descriptor systems. The case of parameters with unbounded variation rates is addressed by means of parameter-independent HPLFs while the case of parameters with bounded rates of variation is discussed through parameter-dependent HPLFs. For both cases, necessary and

sufficient conditions for robust admissibility analysis are given whenever the degree of the HPLF is given. Then, using different convexification techniques, LMI admissibility analysis conditions are derived that cover previous findings in the literature. To the best of the authors' knowledge, the results presented in this paper are the first attempt extending the use of HPLFs developed for standard systems to the case of uncertain descriptor systems. The numerical examples show that the proposed approach leads to less conservative admissibility analysis results.

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