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Energetical formulation of size effect law for quasi-brittle fracture

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ABSTRACT

A criterion for quasi-brittle failure based on local quantities is proposed. The propagation of long cracks is controlled by the energy release rate, while the initiation of the crack is shown to be well described by the derivative of the energy release rate with respect to the crack length. An equation based on well-known models for the analysis of the size and boundary effects is then adapted to describe the competition between these two quantities in order to characterize the local failure. The predictions of rupture of pre-cracked specimens of different sizes and materials are compared to the experimental results given in the literature. The generality of the formulation for describing the crack propagation mechanisms for other defects, such as v-notches and holes, is finally discussed.

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1. Introduction

The stress and energy release are basic elements found in most rupture criteria where the failure is characterized by the initiation and propagation of cracks. For an intact structure, a simple stress criterion may indicate crack initiation for a given stress state that is defined as the material strength. On the other hand, considering an energy criterion, a large crack may propagate if the energy release rate during its propagation reaches a certain value associated with the fracture energy of the material. Intermediate situations, such as the rupture of a small cracked specimen or structures presenting “imperfections” other than cracks (e.g. heterogeneities, complex shaped notches, etc.) are not well described by either of these two criteria. These difficulties have given rise to different approaches in order to predict the rupture behavior of quasi-brittle materials.

Based on the stress criteria, Critical Distance Theories (CDTs) were initially proposed in the 1930s by Neuber [1] and Peterson [2] using a punctual evaluation of the stress at a certain distance from the crack tip (Point Stress Method) and the average of the stress over a certain length (Average Stress Method), respectively, in order to analyze the fatigue failure of metallic structures. Adopting the same concept, Novozhilov [3] proposed a simple failure criterion based on the average normal stress along the anticipated path of the crack formation. This model has been expanded by Seweryn [4–7] to study both regular and singular stress concentrations under mode I or mixed mode loading.

Leguillon [8] proposed a criterion for failure initiation at a sharp v-notch under mode I loading, which requires the stress condition and energy condition to be fulfilled simultaneously. This criterion was compared with several known failure initiation criteria and validated in [8,9] for mode I loading. It was improved by Leguillon and Yosibash [10] by introducing a correction due to the small notch tip radius, later extended by Yosibash et al. [11] to mixed mode loading and validated

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Nomenclature

a, a_t	crack length and transition crack length
a_f, a_{vr}	propagation length scales
A_i	geometrical correction factor
B	positive dimensionless factor in the size effect law
c	v-notch depth
$dG/da, \max dG/da$	derivative of the energy release rate and its critical value
D	diameter of the circular hole
E	Young's modulus
f_t	material tensile strength
$F_0, F_N, F_{\max}$	applied load, failure load, and maximum failure load
G_0, G_c	energy release rate and fracture energy
h	height of the beam
K_0, K_c	stress intensity factor and fracture toughness
K_0^N, K_c^N	generalized stress intensity factor and its critical value
l_{ch}	Irwin's characteristic length
L	span of the beam
r	positive dimensionless factor in the proposed model
$\bar{r}$	positive dimensionless factor in the size effect law
R	radius of the circular hole
t	thickness of the specimen
w	width of the plate
W, W_0	size and transition size of the structure
γ	notch opening angle
η	dimensionless parameter for the energy release rate of a short crack emanating from the v-notch tip
λ	order of stress singularity
Λ	dimensionless parameter for the stress intensity factor of a short crack emanating from the v-notch tip
ν	Poisson ratio
ξ	ratio of K_0^N and F_0
$\sigma_0, \sigma_N, \sigma_{\max}$	applied stress, nominal strength, and maximum failure stress
$\sigma_N^{\text{strength}}$	nominal strength predicted by the strength mechanism

by experimental observations. The concept of Finite Fracture Mechanics was used in Leguillon's criterion, which assumes the instantaneous formation of cracks of finite size at initiation [12]. Instead of the point-wise stress criterion adopted in Leguillon's criterion, Cornetti et al. [13] proposed a similar criterion, based on the evaluation of stresses prior to fracture averaged over the crack. These coupled criterion allows for the general analysis of arbitrary stress concentrations [14–16], and has been used by many researchers to establish general failure criteria for a wide range of engineering problems in the last 14 years [12].

The cohesive zone models [17–19] simulate the damage that occurs in the process zone located ahead of the crack tip. This approach, which involves nonlinear constitutive laws that are described by a displacement jump and corresponding traction along the interfaces, provides a phenomenological model with which to simulate complex fracture behavior, such as crack nucleation, initiation and propagation [20]. The extension of the classical cohesive model to quasi-brittle materials usually shows fractal patterns in the failure process. This fractal approach leads to a scale-invariant cohesive crack model which is able to predict the size effects even in tests where the classical approach fails, e.g. the direct tension test [21].

Most of the methods are associated with a length scale that is usually proportional to Irwin's [22] characteristic length l_{ch} , which depends on the properties of the material, such as the stiffness, strength and fracture toughness. However, for certain building materials (such as concrete, rocks, and some types of ceramics), the value of the characteristic length scale can become too large when compared to the specimen size, and sometimes can even exceed it, which makes the direct implementation of these approaches impossible [23]. In the discrete element methods, the materials are organized into assemblies of particles in contact. Initially developed by Cundall and Strack [24] for modelling granular and particulate systems, these methods were further adapted to study the fracture of quasi-brittle materials, such as concrete [25,26] and rocks [27]. Despite the simple (and physical) local point of view of ruptures, the intrinsic scale effects related to the particle size and characteristic length l_{ch} also affect the response of the model to quasi-brittle rupture [28–30].

This paper presents an alternative description of quasi-brittle failure based only on local quantities that are related to the energy release rate G . Hence, the size [31,32] and boundary effects [33–35] that are observed in the experiments are simply analyzed through the combination of mechanisms related to the strength and fracture toughness of the material, thereby avoiding some of the nonphysical aspects associated with most of the existing models. In Section 2, the notions of the size and boundary effects are presented. Based on the existing expressions used to describe these types of experimental evidence,

the two local quantities are related, which gives rise to the model presented in Section 3. Section 4 presents a comparison of the model with the experimental results in the literature in the opening mode of various cracks: cracked three-point bending beams, v-notched tensile and bending samples, and plates with a circular hole in tension. Finally, the conclusions and perspectives of the present work are presented.

2. Size and boundary effects in quasi-brittle rupture

On a small scale, quasi-brittle materials obey the strength theory that is characterized by the material strength f_t , while on a large scale, they conform to Linear Elastic Fracture Mechanics (LEFM) and are characterized by the fracture energy G_c . The combination of f_t and G_c yields Irwin's [22] characteristic length $l_{ch} = G_c E / f_t^2$, where E is the Young's modulus, and divides the small and large scales [32]. Based on an approximate energy release analysis, the following Type 2 Size Effect Law (SEL) was derived [31] for structures that fail after large stable crack growth (like in Fig. 1a):

$$\sigma_N = B f_t \left[1 + \left(\frac{W}{W_0} \right)^{\bar{r}} \right]^{-1/2\bar{r}}, \quad (1)$$

where σ_N is the nominal strength, $\bar{r}$ and B are positive dimensionless factors (B depends on the geometry of the structure), W represents the size of the structure, and W_0 is the transition size at which the failure laws based on the material strength and LEFM intersect, as shown in Fig. 1b.

Hu and Wittmann [33–35] investigated the size effect on the fracture properties of a large plate with a finite crack (Fig. 1c) and proposed an adaptation of Eq. (1) to describe the dependence of the specimen strength σ_N on the crack length a :

$$\sigma_N = f_t \left[1 + \left(\frac{a}{a_t} \right) \right]^{-1/2}, \quad (2)$$

Fig. 1. (a) Similar cracked structures, (b) the size effect law linking the failure mechanisms of the material strength and LEFM (modified from [32]), (c) a crack in a large plate, and (d) its nominal strength σ_N as a function of the crack length a (modified from [35]).

where a_t is defined by the intersection of the maximum tensile stress and the LFM criteria, and is referred to as the transition crack size $a_t = l_{ch}/[A_1^2(a/w)\pi]$, where A_1 is a coefficient that depends on the geometry of the sample and size of the crack a (for an infinite plate, $A_1(a/w \rightarrow 0) = 1.12$ [35]). In the case of the large plate, the crack size a represents the distance from the crack-tip to the front free surface of the specimen. Eq. (2) indicates that when the distance of the crack-tip to the front boundary is comparable to a_t , the specimen strength will experience the boundary effect. When the crack-tip is very far away from the front boundary, the LFM criterion prevails and the specimen strength will be determined primarily by its fracture energy G_c (see Fig. 1d) [35].

Both equations allow the prediction of a failure of a structure while taking into account the size and boundary effects based on the nominal strength σ_N (a non-local quantity) and structural information, such as the crack size a . In the next section, a generalization of Eq. (2) is proposed that is only based on the material parameters and local quantities.

3. Local approach

3.1. Model

A more general form of Eq. (2) can be used to provide a more concise representation of the failure behavior:

$$\sigma_N = f_t \left[1 + \left(\frac{a}{a_t} \right)^r \right]^{-1/2r}, \quad (3)$$

where r is a parameter that describes the transition between a failure that is defined by the material strength ($a \rightarrow 0$) and one defined by LFM ($a \gg a_t$). The effect of this parameter is presented in Fig. 2a and traduces some complex consequences of the microstructure of the material such as granulometry and voids. Lower values of r lead to smoother transitions between the two mechanisms, which are related to lower rupture values for cracked structures. As shown in detail in Appendix A, the rupture of a quasi-brittle material (represented by Eq. (3)) can be rewritten as the contribution of the two mechanisms:

$$\left(\frac{\sigma_N}{f_t} \right)^{2r} + \left(\frac{G}{G_c} \right)^r = 1. \quad (4)$$

The ratio σ_N/f_t denotes the activation of the material strength, while G/G_c is related to the fracture mechanics.

The non-local quantity $(\sigma_N/f_t)^2$ can be estimated by the ratio between the derivative of the energy release rate with respect to the crack length dG/da and its maximum value $\max dG/da$, as discussed in Appendix B. Then, the failure criterion (Eq. (4)) may be finally rewritten as

$$\left(\frac{dG/da}{\max dG/da} \right)^r + \left(\frac{G}{G_c} \right)^r = 1, \quad (5)$$

which is a function of quantities obtained exclusively at the crack tip and the parameters of the material. The value of $\max dG/da$ can be obtained experimentally from a sample without cracks or simply determined based on the tensile strength f_t and Young's modulus E of the material: $\max dG/da = (1.12f_t)^2\pi/E$ (see Appendix B).

The values of $[(dG/da)/(\max dG/da)]^r$ and $(G/G_c)^r$ depend on the crack size, as shown in Fig. 2b. The ratio $[(dG/da)/(\max dG/da)]^r$ dominates the failure behavior for very small cracks ($a \rightarrow 0$), which is in the region where fracture mechanics is not adapted for quasi-brittle materials; however its effects diminish for longer cracks. By definition, Eq. (5) predicts the initiation of a crack under nominal stresses reaching the tensile strength f_t and crack propagation $G = G_c$ for long cracks. The transition between these two mechanisms occurs at $a = a_t$, which is the transition crack size.

Fig. 2. (a) Nominal strength σ_N as a function of crack length a for different r , and (b) evolution of the ratios $(G/G_c)^r$ and $[(dG/da)/(\max dG/da)]^r$ as functions of the crack length a/a_t ($r = 1$).

The main advantage of Eq. (5) is that it is independent of geometrical parameters. The values of G and its derivative dG/da can be quantified for defects that induce stress singularities, such as cracks or v-notches, or do not induce stress singularities, such as holes, as shown in Section 4.

3.2. Failure load for elastic structures and finite length

The relation represented by Eq. (5) is defined for a failure load F_N . If a different load F_0 is applied, the equation becomes

$$\left(\frac{dG_0/da}{\max dG/da} \right)^r + \left(\frac{G_0}{G_c} \right)^r = \mu_0, \quad (6)$$

where G_0 and dG_0/da are the corresponding energy release rate and its derivative at the crack tip under the loading F_0 . Considering that F_0 is a fraction of the failure loading ($F_N = \rho F_0$), by linearity, $G = \rho^2 G_0$ and $dG/da = \rho^2 dG_0/da$. The value of $\rho = \mu_0^{-1/(2r)}$ can be obtained by substituting the relation for G , dG/da , and Eq. (6) into Eq. (5). Finally, the failure load F_N can be calculated based on the results of any given load F_0 based on the expression

$$F_N = \mu_0^{-1/(2r)} F_0. \quad (7)$$

4. Comparison to experimental results from the literature

The failure criterion proposed in Eq. (5) was compared to the experimental results given in the following section. The tensile strength f_t and fracture energy G_c adopted in the predictions are associated to measured values. However, the parameter r is adjusted in order to minimize the error between predictions and experimental results. The size and boundary effects were clearly visible on the three-point bending tests presented first. The generality of the proposed formulation was then analyzed using the results of the rupture tests of v-notched samples in different configurations. Finally, the crack initiation under a local stress gradient was verified in the tensile tests of samples with circular holes of different sizes.

4.1. Cracked Three-point Bending (C-TPB) tests

4.1.1. Evaluation of G and dG/da

The stress intensity factor K_0 of the C-TPB specimen can be directly calculated by

$$K_0 = \frac{3F_0 L}{2h^2 t} \sqrt{\pi a} A_2, \quad (8)$$

where F_0 is the applied load, L , h , and t are the span, height, and thickness of the beam respectively, and a is the crack length (see Fig. 3a). The empirical formula for the correction factor A_2 was derived by Tada [36] for any crack to height ratio a/h :

$$A_2 = \sqrt{\frac{2h}{\pi a} \tan \frac{\pi a}{2h} \frac{0.923 + 0.199(1 - \sin \frac{\pi a}{2h})^4}{\cos \frac{\pi a}{2h}}}. \quad (9)$$

The value of the energy release rate was obtained from the LFM relation for plane stress $G_0 = K_0^2/E$, and dG_0/da was calculated by deriving G_0 with respect to the crack size a . These values are related to the material fracture energy G_c and material strength f_t , as shown in Section 3.2, and characterize the failure of the structure.

Fig. 3. (a) Three-point bending geometry and (b) the nominal strength σ_N versus the crack length a for C-TPB specimens [37].

4.1.2. Birchall's experiments

Fig. 3b illustrates the nominal strength results ($\sigma_N = 1.5F_N L / (h^2 t)$) predicted by the proposed model and compared to the test results of the C-TPB specimens tested by Birchall et al. [37] that had a span to height ratio of $L/h = 4$ and were made of Portland cement paste. In the figure, the model results were obtained for a beam with dimensions of $h = 1$ m, $t = 1$ m, and $L = 4$ m. The material parameters were measured by Birchall et al. [37]. The mean tensile strength $f_t = 10.5$ MPa and mean fracture energy $G_c = 23.8$ N/m were calculated by measuring the failure loads of uncracked and cracked specimens, respectively, which resulted in a transition crack size of $a_t = 1.1$ mm. The transition between the failure mechanism associated with the material strength and that due to the fracture mechanics (visible in Fig. 3b) is quite sharply delineated, which is well described by a high value of the parameter r ($r = 10$ in this case).

4.1.3. Karihaloo's experiments

The C-TPB tests performed by Karihaloo et al. [38] were conducted on cracked beams made of hardened cement paste with a span to height ratio of $L/h = 4$. The crack to height ratios a/h of the specimens were 0.1, 0.3, and 0.5, and four beams were tested for each value. The heights of the beams were 50 mm, 100 mm, and 200 mm, and all had the same thickness of 100 mm). The mechanical properties of the material were measured on several specimens using standard procedures. The mean values of the tensile strength and Young's modulus were $f_t = 3.53$ MPa and $E = 20.8$ GPa, respectively. The value of the fracture energy $G_c = 13.5$ N/m was calculated using Eq. (8) based on the measured failure load of the largest specimen (with a height of 200 mm), which resulted in a transition crack size of $a_t = 5.72$ mm.

The predicted failure loads given by the proposed model and Bažant's Type 2 Size Effect Law (Eq. (1)), and test results are plotted in Fig. 4a as a function of the crack to height ratio a/h . In these example, the value of $r = 1$ seems adequate, which indicates a smooth transition between the mechanisms of failure based on fracture mechanics and material strength. This parameter is important considering the range of crack sizes a ($5 \text{ mm} < a < 100 \text{ mm}$). The model parameters of Type 2 SEL $G_c = 18.1$ N/m and the length scale $c_f = 7.2$ mm were calibrated for $a/h = 0.3$ [39]. Bf_t and transition size W_0 in Eq. (1) are calculated respectively for each crack to height ratio based on the given fracture energy G_c and length scale parameter c_f .

4.1.4. Higgins' experiments

Higgins and Bailey [40] tested C-TPB specimens of a hardened cement paste with heights h that varied from 5 mm to 110 mm, and a crack to height ratio a/h that varied from 0.02 to 0.5. The span to height ratio was equal to $L/h = 5$ for all beams, and the thickness was equal to $t = 25$ mm. Fig. 4b shows the comparison between the test results and the predicted failure loads obtained by the proposed model and by the Boundary Effect Model (BEM) [35] as functions of the crack to height ratio a/h . The proposed model and boundary effect model are both able to characterize size and boundary effects in this example. The mechanical properties obtained by optimizing the rupture model (for $r = 0.5$) were: tensile strength $f_t = 12$ MPa and fracture toughness $\sqrt{G_c E} = 0.83 \text{ MPa}\sqrt{\text{m}}$ (the Young's modulus E was not provided), resulting in a transition crack size $a_t = 1.21$ mm. These results are consistent with the values obtained by the authors. The highest value of strength obtained in direct tensile tests on uncracked specimens was 12.5 MPa, associated with an estimated fracture toughness of $0.8 \text{ MPa}\sqrt{\text{m}}$. The material parameters adopted by Duan et al. [35] in BEM model were: tensile strength $f_t = 10.29$ MPa and fracture toughness $K_c = 0.65 \text{ MPa}\sqrt{\text{m}}$.

4.2. V-Notched tests

The results for the failure of the C-TPB specimens (Section 4.1) indicates the ability of Eq. (5) to locally characterize the strength of structures displaying cracks of different sizes. In the following section, the failure of structures associated with

Fig. 4. Failure loads F_N versus crack to height ratio a/h for various specimen sizes in the experiments of (a) Karihaloo et al. [38], and (b) Higgins and Bailey [40]. The error bars in (a) indicate the standard deviations of the experimental results.

other imperfections that induce singular stresses is analyzed. Samples with v-notches, as in Fig. 5, are used as a reference due the dependence of the stress field on the opening angle.

4.2.1. Evaluation of G and dG/da

The vertical stress along the x -axis (see Fig. 5) in the vicinity of the notch tip can be expressed as $\sigma_y(x) = (K_0^N)/(2\pi x)^{1-\lambda}$ [14], where λ is the order of the stress singularity, which depends on the notch opening angle γ . λ is equal to $1/2$ for $\gamma = 0^\circ$ (crack) and 1 for $\gamma = \pi$ (straight edge), and K_0^N is the Generalized Stress Intensity Factor (GSIF) whose physical dimensions depend on the notch opening angle γ .

In mode I, the value of the energy release rate for a short crack emanating from the v-notch tip subjected to a given load F_0 can be written as

$$G_0 = \eta \frac{(K_0^N)^2}{E^*} a^{2\lambda-1}, \quad (10)$$

where η is a dimensionless parameter that depends only on the notch opening angle γ [41]; in plane stress $E^* = E$ and plane strain $E^* = E/(1 - \nu^2)$. The value of $dG_0/da = \eta(2\lambda - 1)(K_0^N)^2 a^{2\lambda-2}/E^*$ is obtained by derivation of Eq. (10) with respect to the emanated crack length a . The values of λ , η , and K_0^N are presented in Appendix C for v-notched plates and beams.

4.2.2. Failure identification

At the notch tip ($a \rightarrow 0$), the value of the energy release rate G_0 tends to zero for any singularity weaker than a crack ($1/2 < \lambda < 1$) [12], as indicated by Eq. (10), while its derivative dG_0/da tends to infinity. According to Eqs. (6) and (7), in these conditions, the failure load F_N tends to zero, as shown in Fig. 6a. The propagation of the crack induces an increase of G_0 and a decrease of dG_0/da , which leads to the rapid growth of the F_N value. For very short propagation lengths, the rupture is dominated by the strength mechanism $\sigma_N^{\text{strength}} = \sigma_0 \sqrt{\max dG/da / (dG_0/da)}$, as shown in Fig. 6b. After a certain length, the fracture toughness of the material is mobilized and the rupture behavior follows the prediction of the LEFM, and depends only on the energy release rate. During the transition between the two mechanisms, a maximum value of the failure load $F_{\max}$ is observed, which defines the failure limit of the sample.

4.2.3. Seweryn's experiments

Seweryn et al. [6] carried out tensile experiments using symmetric v-notched polymethyl methacrylate (PMMA) specimens with the following mechanical properties: Young's modulus $E = 3.3$ GPa, Poisson ratio $\nu = 0.35$, tensile strength $f_t = 102.8$ MPa, and plane stress fracture energy $G_c = 437.82$ N/m. The fracture energy G_c was calculated by Seweryn et al. based on the relation between the fracture toughness and critical generalized stress intensity factor K_c^N for notch opening angles $\gamma = 20^\circ, 40^\circ$, and 60° [7]. Samples with v-notch opening angles γ of $20^\circ, 40^\circ, 60^\circ, 80^\circ$, and 100° were studied, and their dimensions are indicated in Fig. 7a with a thickness of 5 mm. The results for $\gamma = 180^\circ$ were based on the tensile strength f_t as determined with a sample presenting a double semicircular indentation.

The predicted failure loads $F_{\max}$ with different notch opening angles are compared to the experimental results in Fig. 7b. A more detailed description of the behavior between $100^\circ < \gamma < 180^\circ$ is provided by the predictions of the strain energy

Fig. 5. V-notched plate under tension loading and the corresponding emanating crack.

Fig. 6. Failure load F_N during crack extension and the definition of the maximum failure load F_{max} for v-notches in (a) linear, and (b) semi-log scales.

Fig. 7. (a) Failure loads F_{max} for different notch opening angles γ . (b) The model prediction is compared to the results of Refs. [6,7].

release criterion [7], which are also presented in Fig. 7b. Good agreement with the complete set of results was observed for $r = 4$.

4.2.4. Dunn's experiments

Dunn et al. [42] performed a series of three-point bending tests on notched specimens (N-TPB) in polymethyl methacrylate (PMMA) with dimensions of $L = 76.2$ mm, $t = 12.7$ mm, and $h = 17.8$ mm, as indicated in Fig. 8a. The tests were performed under plane strain conditions following the procedures outlined in ASTM E-399. The specimens were machined with three different notch opening angles γ : 60° , 90° , and 120° . For each notch opening angle, four notch depths c were used: 1.78 mm, 3.56 mm, 5.33 mm, and 7.11 mm. This choice of dimensions resulted in notch depth to height ratios c/h of

Fig. 8. (a) N-TPB specimen and (b) model predictions of the failure load F_{max} compared to the experimental results of Ref. [42] as a function of the notch to height ratio c/h for various notch opening angles γ .

0.1, 0.2, 0.3, and 0.4, respectively. The PMMA had a Young's modulus $E = 2.3$ GPa, poisson ratio $\nu = 0.36$. The average failure stress $f_t = 124$ MPa was based on the bending tests of unnotched specimens. The fracture toughness $K_c = 1.25$ MPa $\sqrt{\text{m}}$ adopted in the model predictions was calculated based on the relation between this quantity and the critical generalized stress intensity factor for the notch opening angle $\gamma = 60^\circ$, as proposed by Seweryn et al. This value was expected to be more precise than the results obtained from cracked three-point bending (C-TPB) specimens ($K_c = 1.02$ MPa $\sqrt{\text{m}}$ with a standard deviation of 0.12 MPa $\sqrt{\text{m}}$) [4].

The comparison between the predicted failure load $F_{\max}$ (for $r = 4$) and the experimental data is presented in Fig. 8b for different notch opening angles γ .

4.3. Crack initiation in a finite width plate with a circular hole

4.3.1. Geometry and failure mechanism

Green et al. [43] performed open-hole tensile tests using the quasi-isotropic carbon-epoxy laminate IM7-8552 with stacking sequence [45/90/−45/0]4s (for laminates with a nominal thickness of 4 mm). Specimens presenting the same aspect ratio based on the hole diameters D (3.175 mm, 6.35 mm, 12.7 mm, and 25.4 mm) were adopted (for widths $w = 5D$ and lengths $20D$, as shown Fig. 9a). The tensile strength $f_t = 929$ MPa was measured in the unnotched tests, and the fracture toughness $\sqrt{G_c E} = 42.3$ MPa $\sqrt{\text{m}}$ was estimated by Camanho et al. [15].

A post-rupture analysis showed that all specimens with the adopted stacking sequence failed due to fiber failure, which indicates that the effect of the laminate stiffness reduction caused by the progressive matrix failures on the ultimate strength of the laminate was insignificant. Hence, the rupture of these uncracked samples depended on the crack initiation and subsequent propagation thereof due to the stress concentration at the lateral edges of the holes. The failure scheme adopted in this case is presented in Fig. 9b, and is consistent with the failure pattern observed experimentally by the authors.

4.3.2. Evaluation of G_0 and dG_0/da

The energy release rate corresponding to two symmetric cracks emanating from a plate with a central circular hole of radius R is given by [44] for an isotropic plate as

$$G_0 = \frac{\sigma_0^2 \pi a}{E} A_3^2 A_4^2, \quad (11)$$

where σ_0 is the applied stress, a is the extended crack length from the hole, and A_3 is the boundary correction factor for the circular hole. The equation for A_3 is

$$A_3 = 1 + 0.358 \left(\frac{R}{d} \right) + 1.425 \left(\frac{R}{d} \right)^2 - 1.578 \left(\frac{R}{d} \right)^3 + 2.156 \left(\frac{R}{d} \right)^4, \quad (12)$$

with $d = a + R$ as indicated in Fig. 9b. The quantity A_4 is the finite width correction factor and is equal to

Fig. 9. (a) Geometry of the plates with circular holes (modified from the illustration by [43]), and (b) the corresponding failure mechanism (two symmetric cracks emanating from two sides of the hole).

$$A_4 = \sqrt{\sec\left(\frac{\pi R}{w}\right) \sec\left(\frac{\pi d}{w}\right)}. \quad (13)$$

The value of dG/da is obtained by derivation of the expression for G (Eq. (11)) with respect to a .

4.3.3. Failure identification

When the hole diameter is sufficiently large, two limit failure mechanisms may be observed when considering the rupture of the plate: one associated with the material strength ($d \rightarrow R, G \rightarrow 0$), and another associated with the fracture mechanics ($d \gg R$). The hole induces a stress concentration of $3\sigma_0$ in an infinite plate, which may be inferior to the tensile strength of the material f_t . Hence, in the absence of any cracks, the failure may occur for $\sigma_N = f_t/3$, which defines the initiation of the crack. On the other hand, after a relatively long crack propagation, the structure may behave as a simple middle-cracked plate of crack length $2(R+a)$ and be governed solely by the fracture mechanics ($G \leq G_c$). The transition between these two mechanisms is presented in Fig. 10a. The individual prediction of the strength mechanism $\sigma_N^{\text{strength}} = \sigma_0 \sqrt{\max dG/da / (dG_0/da)}$ tends to be almost independent of the extended crack size a , as is expected for a quasi-brittle material, until the transition to LEFM behavior. In a finite width plate, the stress concentration induced by the hole tends to be bigger due to the reduction of the cross section of the sample. For a width to diameter ratio $w/D = 5$, this value is equal to $3\sigma_0 \times A_4(a=0) = 3.15\sigma_0$.

For small diameters, the strength results [45,46] are affected by size effects [47]. This behavior is described by Eq. (5) as an increase in the load limit of the plate during crack initiation, following the trend of $\sigma_N^{\text{strength}}$ up to a maximum value before decreasing following the LEFM behavior, as shown in Fig. 10b. This apparent nonphysical behavior is probably an “artifact” due to the simplicity of Eqs. (3) and (5), and will be discussed in Section 5. Consider these size effects would theoretically prevent this increase in the sample strength. One may then predict a behavior similar to that shown in Fig. 10a for any hole size, which is a stable failure stress for small cracks, and is followed by a decrease as predicted by LEFM. However, it seems much simpler to define the failure load based on the maximum value $\sigma_{\max} = \max \sigma_N$ for all cases. The error associated with this simplification will be reduced in practical situations that concern relatively small crack sizes $a < a_t$ (e.g. $a_t \approx 0.5$ mm for the laminated sample used in this example).

4.3.4. Comparison to experimental results

The predictions of the proposed model for $r = 15$ are compared to experimental results and other model predictions in Fig. 11. The theoretical results coincide with the predictions using the average stress method given by Green et al. [43] for the failure stress $\sigma_{\max}$, as shown in Fig. 11a. For the smallest hole diameters D (3.175 mm and 6.35 mm), the failure mechanism was associated with the fiber pull-out across the width, which may explain the relative deviation of the results. It should be noted that this also happens with respect to the average stress criterion. The predictions of finite fracture mechanics given by Camanho et al. [15] also present an offset between the results of the smallest hole and the others.

Holes with bigger diameters induce a failure stress $\sigma_{\max}$ near $f_t/3$, as is expected due to the stress concentration of approximately $3\sigma_0$ at the edge of the holes. Fig. 11b illustrates the difference between the predictions of the proposed model and the size effect law. Type 2 SEL fits well the test results, but predicts a failure stress $\sigma_{\max}$ which fades to zero for large holes. The extended size effect law [48,49] corrects this trend for the holed specimens, which is also the case of the proposed model.

5. Failure process and length scales

The criterion represented by Eq. (5) characterizes the rupture as continuous process. It defines the crack initiation and its propagation based on energetical quantities. In some cases, such as structures presenting weak singularities (e.g. v-notches)

Fig. 10. Nominal strength σ_N in a plate (a) with a big hole, and (b) with a small hole as function of the extended crack size a .

Fig. 11. Prediction of the failure stress σ_{max} for various hole diameters D compared to the experimental [43] and model-based results. (a) Comparison between the proposed model, average stress criterion and finite fracture mechanics [15]. The insets depict the rupture patterns for different hole diameters. (b) Comparison between the proposed model and two versions of the size effect law (SEL).

or stress concentrations (e.g. holes), the maximum value of the load F_N was obtained after a propagation length a_f . At this length scale, F_N is maximized, which means that a_f mathematically minimizes the quantity μ_0 (Eq. (6)) with respect to the crack length a . After some algebraic work, the equation $d\mu_0/da = 0$ can be rewritten as

$$\left(\frac{-(G_0)^{r-1}}{(dG_0/da)^{r-2} d^2 G_0/da^2} \right)^{1/r} = \frac{G_c}{\max dG/da} = \frac{l_{ch}}{1.12^2 \pi}, \quad (14)$$

where $l_{ch} = G_c E / f_t^2$ is the characteristic length of the material. The length a_f , solution of Eq. (14), is easily obtained for a v-notch plate ($a_f = 0.254 l_{ch} (2 - 2\lambda)^{1/r} (2\lambda - 1)^{(1-1/r)}$). In Fig. 12a, the value of a_f/l_{ch} is plotted as a function of the notch opening angle γ for $r = 4$. The ratio a_f/l_{ch} can be obtained numerically for a finite width plate with a circular hole. In Fig. 12b, a_f/l_{ch} is presented for different hole diameters D ($r = 5$ and width $w = 5D$).

The analysis of the length $a_{vr} = (-G)^{r-1} / ((dG/da)^{r-2} d^2 G/da^2)^{1/r}$ (based on the left-hand side of Eq. (14)) can technically remedy the nonphysical increase of the failure load observed during crack initiation in v-notched (Fig. 6) or holed plates (Fig. 10). Only if $a_{vr} \geq l_{ch}/(1.12^2 \pi)$, which corresponds to $a \geq a_f$, the failure load should be considered as physically consistent. Otherwise, the propagated crack a would only present a mathematical meaning on the crack initiation process. Consider a finite crack length absolutely bigger than a_f for the calculations, similar to finite fracture mechanics approaches [12,13,15] can also prevent any increase on the failure load.

Despite all possible approaches to avoid this apparent problem of the increase of the failure load, it should be noted that the length scale a_f is directly related to the characteristic length l_{ch} , usually associated to the size of the fracture process zone (FPZ), and the parameter r . The FPZ is an important element in many models [5,8,13], physically associated to a region containing micro-cracks near the crack tip. In small scales ($a \rightarrow 0$), the size effect behavior is particular and alternative models are proposed under these conditions [50,51]. The improvement of Eq. (5) based on these effects for $a_{vr} \leq l_{ch}/(1.12^2 \pi)$ may provide a finer and more realistic description of the crack initiation.

Fig. 12. Ratio a_f/l_{ch} (a) versus notch opening angle γ for a v-notched plate and (b) as a function of the hole diameter D for a holed plate ($l_{ch} = 2.07$ mm).

6. Conclusions

The local model proposed in this paper consistently reflects the main rupture trends from crack initiation to crack propagation based on its agreement with the experimental results for different geometries. In all analyzed cases, the failure was induced by an opening mode wherein the value of the energy release rate G controlled the propagation of the crack, and its derivative with respect to the crack size dG/da controlled the crack initiation. The competition between these two quantities was related to the material parameters, namely, the fracture energy G_c , tensile strength f_t , and parameter r , which characterizes the transition between the crack initiation and propagation states. Despite the clear physical sense of r , only further testing with the same material and completely different geometries would definitively allow the association of this parameter exclusively to this material.

The principal feature of the model is to identify the effects of the nominal stress (a non-local quantity) through the evolution of the energy release rate G (in a local sense) specifically at the initiation of the crack. This leads to an alternative description of the rupture, which allows a relatively simple analysis of structures with any size of crack, or no cracks at all. For complex geometries and/or boundary conditions, a numerical implementation of the present formulation may produce convergent results with respect to the mesh sizes, depending only on the precision of the G and dG/da values. The scope of LEFM is extended to a certain extent, considering that neither plasticity nor damage is necessarily associated with the model to describe crack initiation.

Size effects caused by stress concentration (e.g. at the edges of the holes), were theoretically quantified using the value of dG/da . Despite the simplicity of Eq. (5), the equation provides good insight into the competition between the failure mechanisms at a local level. Information regarding this point of view associated to the existing background may contribute to the improvement of rupture analysis.

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Appendix A. Failure mechanisms

The crack size dependence in Eq. (3) can be rewritten by considering the notion of the energy release rate. In an elastic plate under tension (Fig. 1c), the energy release rate (in mode I) can be expressed as

$$G = \frac{[A_1(a/w)\sigma_N]^2 \pi a}{E}, \quad (\text{A.1})$$

which is a fundamental quantity used to describe the propagation of cracks in fracture mechanics when compared to the material fracture energy G_c , and $A_1(a/w)$ is a coefficient that depends on the geometry of the element and the size of the crack a .

The crack length a can be obtained by

$$a = \frac{GE}{\pi[A_1(a/w)\sigma_N]^2}. \quad (\text{A.2})$$

The transition crack length a_t can be defined based on the value of a (Eq. (A.2)) when $G = G_c$ (fracture energy) and $\sigma_N = f_t$ (material strength):

$$a_t = \frac{G_c E}{\pi[A_1(a/w)f_t]^2}. \quad (\text{A.3})$$

After substituting Eqs. (A.2) and (A.3) into Eq. (3) and some algebraic work, one obtains

$$\left(\frac{\sigma_N}{f_t}\right)^{2r} + \left(\frac{G}{G_c}\right)^r = 1. \quad (\text{A.4})$$

Appendix B. Local evaluation of the nominal strength σ_N

Considering an elastic plate under tension, the rate of variation of G (see Eq. (A.1)) for an infinitesimal propagation of the crack da can be written as

$$\frac{dG}{da} = \frac{[A_1(a/w)\sigma_N]^2 \pi}{E} + \frac{2A_1(a/w)\sigma_N \pi}{E} \frac{dA_1(a/w)}{da} a. \quad (\text{B.1})$$

For very small cracks ($a \rightarrow 0$), the second term of Eq. (B.1) vanishes, and dG/da contains useful information about the nominal stress that is inducing G :

$$\frac{dG}{da} = \frac{[A_1(a/w \rightarrow 0)\sigma_N]^2 \pi}{E}, \quad (\text{B.2})$$

where $A_1(a/w \rightarrow 0) = 1.12$ for an edge crack [35].

The maximum value of dG/da may be obtained for an intact plate when tested at its tensile limit ($\sigma_N = f_t$):

$$\max \frac{dG}{da} = \frac{[A_1(a/w \rightarrow 0)f_t]^2 \pi}{E} = \frac{(1.12f_t)^2 \pi}{E}. \quad (\text{B.3})$$

Hence, the ratio between dG/da and its maximum value can be used to replace the ratio σ_N/f_t in Eq. (4):

$$\frac{dG/da}{\max dG/da} \approx \left(\frac{\sigma_N}{f_t} \right)^2. \quad (\text{B.4})$$

Appendix C. Evaluation of the energy release rate G_0 for v-notched samples

The stress intensity factor K_0 corresponding to a short crack emanating from the notch tip depends on the generalized stress intensity factor K_0^N , notch opening angle γ , and crack length a [52]. The relation between the two stress intensities and the crack length is given by [41] for cracks that are much shorter than the notch depth. Which reads

$$K_0 = \Lambda(\gamma) \sqrt{\pi} \frac{K_0^N}{(2\pi)^{1-\lambda}} a^{\lambda-1/2}, \quad (\text{C.1})$$

where Λ is plotted and tabulated in [41]. It should be noted that there is a difference of $1/(2\pi)^{1-\lambda}$ between Eq. (C.1) and the original equation presented in [41] due to the different formal definition of generalized stress intensity factor.

The substitution of Eq. (C.1) into the well-known Irwin's relation in plane stress ($E^* = E$) and plane strain ($E^* = E/(1 - \nu^2)$) yields:

$$G_0 = \eta \frac{(K_0^N)^2}{E^*} a^{2\lambda-1}, \quad (\text{C.2})$$

for $\eta = \Lambda^2(\gamma) \pi / (2\pi)^{2-2\lambda}$, calculated based on the highly accurate Λ values given by Philipps et al. [41] for the mode I case. The generalized stress intensity factor K_0^N is related to the remote loading F_0 by $K_0^N = \xi F_0$, where the coefficient ξ for different specimens can be obtained numerically. The values of ξ for v-notched plates with different notch opening angles γ are tabulated in [7]. For v-notched beams, ξ values are calculated from the values of $K_c^N / (2\pi)^{1-\lambda}$ and σ_{max} tabulated in [42] ($\xi = K_c^N / [\sigma_{max} 2h^2 t / (3L)]$, where h , t , and L are the height, thickness, and spam of the beams, respectively).

Table C.1 presents the values of λ , ξ , Λ and η for different notch opening angles for v-notched plates and beams.

Table C.1

Parameters λ , ξ , Λ and η for v-notched plates and beams. The values with * were not originally tabulated but have been calculated by linear interpolation.

γ	λ [7,42,53]	ξ of plates (Pa m ^{1-λ} /N) [7]	ξ of beams (Pa m ^{1-λ} /N) [42]	Λ [41]	η
0	0.5	613.3	n/a	1.414	1.000
20	0.5004	616.6	n/a	1.417*	1.005
30	0.5015	n/a	n/a	1.418	1.011
40	0.5035	632.3	n/a	1.414*	1.013
60	0.5122	667.3	2335.5	1.407	1.035
80	0.5304	735.0	n/a	1.389*	1.079
90	0.5445	n/a	2964.6	1.380	1.121
100	0.5628	853.7	n/a	1.362*	1.168
120	0.6157	1057.6	4703.8	1.327	1.347
140	0.6972	1424.6	n/a	1.270*	1.665
150	0.7520	n/a	n/a	1.241	1.944
160	0.8187	2153.5	n/a	1.201*	2.327
180	1	6314.6	n/a	1.120	3.941

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